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# On the Generalized Fractional Convection–Diffusion Equation with an Initial Condition in $\mathbb{R}^n$

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**Abstract:** Time-fractional convection–diffusion equations are significant for their ability to model complex transport phenomena that deviate from classical behavior, with numerous applications in anomalous diffusion, memory effects, and nonlocality. This paper derives, for the first time, a unique series solution to a multiple time-fractional convection–diffusion equation with a non-homogenous source term, based on an inverse operator, a newly-constructed space, and the multivariate Mittag–Leffler function. Several illustrative examples are provided to show the power and simplicity of our main theorems in solving certain fractional convection–diffusions equations. Additionally, we compare these results with solutions obtained using the AI model DeepSeek-R1, highlighting the effectiveness and validity of our proposed methods and main theorems.

**Keywords:** generalized fractional convection–diffusion equation; multivariate Mittag–Leffler function; inverse operator; Fourier transform

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## 1. Introduction

By applying an inverse operator and the multivariate Mittag–Leffler function, we obtain a unique series solution to the following time-fractional convection–diffusion equation with an initial condition and a source term, for constants $a, b, \gamma \in \mathbb{R}$:

$$\begin{cases} {}_c\dfrac{\partial^\alpha}{\partial t^\alpha} W(t,\mathbf{x}) + a\, {}_c\dfrac{\partial^\beta}{\partial t^\beta} W(t,\mathbf{x}) = b \triangle W(t,\mathbf{x}) + \gamma \nabla W(t,\mathbf{x}) + \phi(t,\mathbf{x}), \\ W(0,\mathbf{x}) = \psi(\mathbf{x}), \end{cases} \tag{1}$$

where $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$, $0 < \beta < \alpha \le 1$,

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \quad \nabla = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n},$$

and the partial Liouville–Caputo fractional derivative ${}_c\frac{\partial^\alpha}{\partial t^\alpha}$ of order $0 < \alpha \le 1$ with respect to $t$ is defined as

$$\left( {}_c\frac{\partial^\alpha}{\partial t^\alpha} W \right)(t,\mathbf{x}) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} W'_\tau(\tau,\mathbf{x}) d\tau.$$

We note that $b$ is the diffusion coefficient, $\gamma$ is the velocity field of the fluid, and $\phi$ denotes the sources.

In particular, we have the non-homogeneous convection equation, given below, if $\alpha = 1$ and $a = b = 0$:

$$\begin{cases} \frac{{}_c\partial}{\partial t} W(t, \mathbf{x}) = \gamma \nabla W(t, \mathbf{x}) + \phi(t, \mathbf{x}), \\ W(0, \mathbf{x}) = \psi(\mathbf{x}), \end{cases} \tag{2}$$

and the non-homogeneous heat equation:

$$\begin{cases} \frac{{}_c\partial}{\partial t} W(t, \mathbf{x}) = b \triangle W(t, \mathbf{x}) + \phi(t, \mathbf{x}), \\ W(0, \mathbf{x}) = \psi(\mathbf{x}), \end{cases} \tag{3}$$

if $\alpha = 1,\ a = \gamma = 0$.

Equation (1) is a multi-term time-fractional differential equation that describes the transport of a scalar quantity—such as concentration, temperature, or momentum—in a fluid medium due to the combined effects of convection (bulk motion) and diffusion (random molecular motion). It is widely used in fields such as fluid dynamics, heat transfer, and environmental modeling [1–3]. Moreover, practical applications of the studied fractional differential equation include modeling contaminant transport in soils and groundwater, where heterogeneous pore structures cause trapping, delays, and the slow release of pollutants from binding sites. It is also used to model non-Fourier heat transfer, particularly in materials such as polymers, biological tissues, and porous media. The use of multiple fractional time derivatives of different orders allows for more accurate modeling of complex memory effects and anomalous diffusion behaviors than single-order models. Each fractional derivative term, with an order between 0 and 1, captures a distinct memory kernel characterized by a specific decay rate. When combined, these terms model systems with multiple relaxation times or heterogeneous memory behaviors. Lower-order derivatives (e.g., 0.2) correspond to strong memory effects and slower diffusion, while higher-order derivatives (closer to 1) indicate weaker memory and faster system responses. Together, they enable the modeling of transitional behaviors between different anomalous diffusion regimes. In materials or media with multiple trapping mechanisms, binding sites, or structural heterogeneity, different fractional orders naturally reflect the various types of obstructions or delays experienced by transported particles.

Physical scenarios where multi-term fractional equations are particularly relevant include heat conduction in composite or porous materials and drug diffusion in biological tissues, where complex temporal dynamics and structural variability play a significant role.

Generally speaking, there are many analytic approaches [4] (fractional Green's function, separation of variables, integral transforms, adomian decomposition method, and homotopy analysis method) and numerical methods [2] (finite difference methods, finite element methods, spectral methods, and meshless methods) dealing with fractional partial deferential equations. In 2024, Chertovskih et al. [5] studied an optimal control problem for a stochastic differential equation related to the Fokker–Planck–Kolmogorov equation in the space of probability measures and proposed a rapidly converging numerical algorithm to address the fundamental nature of diffusion processes. In 1997, Tannehill et al. [6] studied numerical schemes (e.g., upwind, SUPG) for convection–dominated flows. In 2008, Roos et al. [7] addressed stabilization techniques for singularly perturbed convection–diffusion equations. In 2007, Lin and Xu [8] considered the numerical resolution for the following time-fractional diffusion equation for $0 \leq \alpha \leq 1$:

$$\frac{{}_c\partial^\alpha}{\partial t^\alpha}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = f(t,x), \ \ 0 < t \leq T, \ \ x \in \Lambda = (0, L)\ T, L > 0,$$

subject to the initial and boundary conditions:

$$u(0,x) = g(x), \ \ x \in \Lambda, \ \ u(t,0) = u(t,L) = 0,$$

and proved that the full discretization is unconditionally stable, and the numerical solution converges to the exact one with the order $O(\triangle t^{2-\alpha} + N^{-m})$, where $\triangle t$, $N$, and $m$ are the time-step size, polynomial degree, and regularity of the exact solution, respectively. However, we believe that applying a numerical approach to Equation (1) presents significant challenges due to the unbounded domain and the presence of complex terms. To use the inverse operator method, it is necessary to represent the inverse operator as an infinite series, which must be well-defined within an appropriate function space.

On the application side, Bear [9] applied convection–diffusion equations to model groundwater flow and contaminant transport. LeVeque [10] discussed finite volume methods for conservation laws, including convection–diffusion systems.

Variants of Equation (1) with a distributed order have been successfully analyzed in various contexts, and it has been shown that these variants can lead to different diffusion regimes. For example, Luchko [11] investigated the maximum principle for several types of generalized time-fractional diffusion equations—including multi-term and distributed-order diffusion equations—and applied it to establish the uniqueness of solutions to various initial-boundary value problems. Additionally, Kilbas et al. [12] constructed a solution to Equation (1) for the case $a = 0$, using the Laplace transform, multivariate Mittag–Leffler functions, and Green's functions. In contrast, our current work employs inverse operators to derive a unique and novel series solution, offering a significantly simpler computational approach that avoids the complex integrals and convolutions typically required when using Green's functions.

In summary, the multi-term time-fractional convection–diffusion equation extends classical transport models by incorporating memory effects and multi-scale dynamics through fractional derivatives of distinct orders. This equation is indispensable for modeling systems in which transport processes exhibit anomalous behavior or long-range temporal dependencies.

Let $\alpha_1, \cdots, \alpha_m, \beta > 0$ and $z_1, \cdots, z_m \in \mathbb{C}$. Then

$$E_{(\alpha_1,\cdots,\alpha_m),\,\beta}(z_1,\cdots,z_m) = \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_m=j} \binom{j}{j_1,\cdots,j_m} \frac{z_1^{j_1}\cdots z_m^{j_m}}{\Gamma(\alpha_1 j_1 + \cdots + \alpha_m j_m + \beta)}$$

is the well-known multivariate Mittag–Leffler function [12–14], which is an entire function on complex plane $\mathbb{C}^m$. When $m = 1$, it reduces to the following two-parameter Mittag–Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta > 0, \ z \in \mathbb{C}.$$

If $\beta = 1$, we obtain the classical Mittag–Leffler function defined by

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad \alpha > 0, \ z \in \mathbb{C}.$$

The partial fractional integral operator $I_t^\alpha$ ($\alpha \geq 0$) is defined as

$$I_t^\alpha W(t,\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} W(\tau,\mathbf{x}) d\tau, \quad t \in \mathbb{R}^+.$$

In particular, $I_t^0 W(t,\mathbf{x}) = W(t,\mathbf{x})$.

The inverse operator method is a powerful tool for studying the uniqueness, existence, and stability of differential equations with various conditions and variable coefficients [15]. In addition, it is also useful for finding analytic solutions to partial fractional-differential equations or partial integro-differential equations. As an example, we present the following theorem to demonstrate applications of inverse operators in fractional partial differential equations, thereby laying the groundwork for Equation (1). In addition, we show that the inverse operator method is also applicable to equations with variable coefficients.

**Theorem 1.** *The following equation with a variable coefficient for all $\alpha_i \geq 0$ ($i = 1, \cdots, n$) and $1 < \alpha \leq 2$ in the space $C([0,1] \times [0,1]^n)$:*

$$\begin{cases} \frac{{}^c\partial^\alpha}{\partial t^\alpha} W(t,\mathbf{x}) + a(\mathbf{x})\ I_1^{\alpha_1} \cdots I_n^{\alpha_n} W(t,\mathbf{x}) = f(t,\mathbf{x}), \ \ (t,\mathbf{x}) \in [0,1] \times [0,1]^n, \\ W(0,\mathbf{x}) = \phi_1(\mathbf{x}), \quad W_t'(0,\mathbf{x}) = \phi_2(\mathbf{x}), \end{cases} \tag{4}$$

*where*

$$I_1^{\alpha_1} \cdots I_n^{\alpha_n} W(t,\mathbf{x}) = \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \cdot \int_0^{x_1} \cdots \int_0^{x_n} (x_1-\tau_1)^{\alpha_1-1} \cdots (x_n-\tau_n)^{\alpha_n-1} W(t,\tau_1,\cdots,\tau_n) d\tau_n \cdots d\tau_1,$$

*has a solution*

$$\begin{aligned} W(t,\mathbf{x}) &= \sum_{k=0}^\infty (-1)^k \big(a(\mathbf{x})\ I_t^\alpha I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k \big(I_t^\alpha f(t,\mathbf{x}) + \phi_1(\mathbf{x}) + \phi_2(\mathbf{x}) t\big) \\ &= \sum_{k=0}^\infty (-1)^k I_t^{\alpha k+\alpha} \big(a(\mathbf{x})\ I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k f(t,\mathbf{x}) \\ &\quad + \sum_{k=0}^\infty (-1)^k \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \big(a(\mathbf{x})\ I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k \phi_1(\mathbf{x}) \\ &\quad + \sum_{k=0}^\infty (-1)^k \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \big(a(\mathbf{x})\ I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k \phi_2(\mathbf{x}). \end{aligned}$$

**Proof.** Applying the operator $I_t^\alpha$ to both sides of Equation (4), we have

$$W(t,\mathbf{x}) - \phi_1(\mathbf{x}) - \phi_2(\mathbf{x})t + a(\mathbf{x})\ I_t^\alpha I_1^{\alpha_1} \cdots I_n^{\alpha_n} W(t,\mathbf{x}) = I_t^\alpha f(t,\mathbf{x}),$$

using the initial conditions given. This implies that

$$\big(1 + a(\mathbf{x})\ I_t^\alpha I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big) W(t,\mathbf{x}) = I_t^\alpha f(t,\mathbf{x}) + \phi_1(\mathbf{x}) + \phi_2(\mathbf{x})t. \tag{5}$$

We are going to show that the inverse operator (unique) of $1 + a(\mathbf{x})\ I_t^\alpha I_1^{\alpha_1} \cdots I_n^{\alpha_n}$ is

$$V_0 = \sum_{k=0}^\infty (-1)^k \big(a(\mathbf{x})\ I_t^\alpha I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k$$

in the space $C([0,1] \times [0,1]^n)$. In fact, for any $f \in C([0,1] \times [0,1]^n)$,

$$\|V_0 f\| \leq \|f\| \sum_{k=0}^{\infty} \|a\|^k \left\|I_t^{\alpha k}\right\| \left\|I_1^{\alpha_1 k}\right\| \cdots \left\|I_n^{\alpha_n k}\right\|$$
$$\leq \|f\| \sum_{k=0}^{\infty} \|a\|^k \frac{1}{\Gamma(\alpha k+1)} \frac{1}{\Gamma(\alpha_1 k+1)} \cdots \frac{1}{\Gamma(\alpha_n k+1)} < +\infty,$$

where
$$\|f\| = \max_{(t,\mathbf{x})\in[0,1]\times[0,1]^n} |f(t,\mathbf{x})|, \quad \|a\| = \max_{\mathbf{x}\in[0,1]^n} |a(\mathbf{x})|.$$

Hence , the operator $V_0$ is well-defined in the space $C([0,1]\times[0,1]^n)$. Furthermore,

$$V_0\big(1 + a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big) = \big(1 + a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big) V_0 = 1 \text{ (identity operator)}.$$

Clearly,

$$V_0\big(1 + a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big) = V_0 + \sum_{k=0}^{\infty} (-1)^k \big(a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^{k+1}$$
$$= 1 + \sum_{k=1}^{\infty} (-1)^k \big(a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^{k} + \sum_{k=0}^{\infty} (-1)^k \big(a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^{k+1}$$
$$= 1 + \sum_{k=0}^{\infty} (-1)^{k+1} \big(a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^{k+1} + \sum_{k=0}^{\infty} (-1)^k \big(a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^{k+1}$$
$$= 1.$$

Similarly, $\big(1 + a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big) V_0 = 1$. Assuming $V_0'$ is another inverse operator. Then

$$\big(1 + a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big) V_0' = 1$$

and applying $V_0$ to both sides of the above implies that $V_0 = V_0'$.

From Equation (5), we derive

$$W(t,\mathbf{x}) = \sum_{k=0}^{\infty} (-1)^k \big(a(\mathbf{x})\ I_t^{\alpha} I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k \big(I_t^{\alpha} f(t,\mathbf{x}) + \phi_1(\mathbf{x}) + \phi_2(\mathbf{x}) t\big)$$
$$= \sum_{k=0}^{\infty} (-1)^k I_t^{\alpha k+\alpha} \big(a(\mathbf{x})\ I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k f(t,\mathbf{x})$$
$$+ \sum_{k=0}^{\infty} (-1)^k \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \big(a(\mathbf{x})\ I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k \phi_1(\mathbf{x})$$
$$+ \sum_{k=0}^{\infty} (-1)^k \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \big(a(\mathbf{x})\ I_1^{\alpha_1} \cdots I_n^{\alpha_n}\big)^k \phi_2(\mathbf{x}),$$

which converges under the norm of the space $C([0,1]\times[0,1]^n)$. This completes the proof. □

Very recently, Li [16] considered the following fractional differential equation using the inverse operator method:

$$\begin{cases} \frac{{}_c\partial^{\alpha}}{\partial t^{\alpha}} u(t,\mathbf{x}) = \triangle_{\lambda_1,\cdots,\lambda_n} u(t,\mathbf{x}) + g(t,\mathbf{x}), \ \ 1 < \alpha \leq 2, \\ u(0,\mathbf{x}) = \phi_1(\mathbf{x}), \ \ u_t'(0,\mathbf{x}) = \phi_2(\mathbf{x}), \ \ (t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^n, \end{cases} \tag{6}$$

where
$$\triangle_{\lambda_1,\cdots,\lambda_n} = \lambda_1 \frac{\partial^2}{\partial x_1^2} + \cdots + \lambda_n \frac{\partial^2}{\partial x_n^2}, \ \text{ all } \lambda_i \text{ are contants,}$$

and obtained

$$u(t,\mathbf{x}) = \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha} \triangle_{\lambda_1,\cdots,\lambda_n}^k g(t,\mathbf{x}) + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \triangle_{\lambda_1,\cdots,\lambda_n}^k \phi_1(\mathbf{x}) + \sum_{k=0}^{\infty} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \triangle_{\lambda_1,\cdots,\lambda_n}^k \phi_2(\mathbf{x}),$$

in a subspace of $C(\mathbb{R}^+ \times \mathbb{R}^n)$.

In particular, if $a = \gamma = 0$, then Equation (1) reduces to

$$\begin{cases} \dfrac{{}_c\partial^\alpha}{\partial t^\alpha} W(t,\mathbf{x}) = b \triangle W(t,\mathbf{x}) + \phi(t,\mathbf{x}), \\ W(0,\mathbf{x}) = \psi(\mathbf{x}), \end{cases}$$

which is similar to Equation (6), except for the difference in the fractional order and the fact that $\triangle_{\lambda_1,\cdots,\lambda_n}$ is a more general operator than $b\triangle$.

The remainder of the paper is structured as follows. Section 2 derives a unique series solution to Equation (1) based on a new space $S$, an inverse operator, and the multivariate Mittag–Leffler function. Section 3 discusses a time-fractional non-homogenous convection equation, which is the particular case for Equation (1), and show that our solution is consistent with the existing integral convolution solution using the Fourier transforms and Taylor's expansion of the two-parameter Mittag–Leffler function. Section 4 works on a time-fractional non-homogenous convection equation obtained from Equation (1) in detail with an illustrative example showing applications of our results. Section 5 proves that our series solution derived in Section 2 coincides with the classical integral solution. Finally. we summarize the entire work in Section 6.

## 2. A Unique Series Solution to Equation (1)

**Theorem 2.** *Let $a, b, \gamma \in \mathbb{R}$ and $0 < \beta < \alpha \leq 1$. Then, Equation (1) has a unique solution:*

$$\begin{aligned} W(t,\mathbf{x}) = & \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} (-a)^{k_1} b^{k_2} \gamma^{k_3} I_t^{\alpha k-\beta k_1+\alpha} \triangle^{k_2} \nabla^{k_3} \phi(t,\mathbf{x}) \\ & + \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} \frac{(-at^{\alpha-\beta})^{k_1} (bt^\alpha)^{k_2} (\gamma t^\alpha)^{k_3}}{\Gamma((\alpha-\beta)k_1+\alpha k_2+\alpha k_3+1)} \triangle^{k_2} \nabla^{k_3} \psi(\mathbf{x}) \\ & + a t^{\alpha-\beta} \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} \\ & \cdot \frac{(-at^{\alpha-\beta})^{k_1} (bt^\alpha)^{k_2} (\gamma t^\alpha)^{k_3}}{\Gamma((\alpha-\beta)k_1+\alpha k_2+\alpha k_3+\alpha-\beta+1)} \triangle^{k_2} \nabla^{k_3} \psi(\mathbf{x}), \end{aligned} \tag{7}$$

*if $\phi$ and $\psi$ in the space $S$ given by*

$$S = \Big\{ f \in C(\mathbb{R}^+ \times \mathbb{R}^n) : \textit{for any non-negative } n\textit{-tuple of integers } (k_1,\cdots,k_n), \; \exists \textit{ a constant } M_f > 0 \textit{ and a positive function } \theta(t,\mathbf{x}) \in C(\mathbb{R}^+ \times \mathbb{R}^n), \textit{ such that } \left| \left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{k_n} f(t,\mathbf{x}) \right| \leq \theta(t,\mathbf{x})\, M_f^{k_1+\cdots+k_n} \Big\}.$$

**Proof.** Applying the operator $I_t^\alpha$ to Equation (1), we obtain

$$W(t,\mathbf{x}) - \psi(\mathbf{x}) + a\, I_t^{\alpha-\beta}(W(t,\mathbf{x}) - \psi(\mathbf{x})) = b\, I_t^\alpha \triangle W(t,\mathbf{x}) + \gamma\, I_t^\alpha \nabla W(t,\mathbf{x}) + I_t^\alpha \phi(t,\mathbf{x}),$$

using the condition $W(0,\mathbf{x}) = \psi(\mathbf{x})$. Hence,

$$\left(1 + a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha} \triangle - \gamma\, I_t^{\alpha} \nabla\right) W(t,\mathbf{x}) = I_t^{\alpha}\phi(t,\mathbf{x}) + \psi(\mathbf{x}) + \frac{a\,\psi(\mathbf{x})\, t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}. \tag{8}$$

Here, we show that a unique inverse operator of $1 + a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha} \triangle - \gamma\, I_t^{\alpha}\nabla$ is

$$\begin{aligned} V &= \sum_{k=0}^{\infty} (-1)^k \left(a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha} \triangle - \gamma\, I_t^{\alpha} \nabla\right)^k \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} a^{k_1} I_t^{(\alpha-\beta)k_1} (-b)^{k_2} I_t^{\alpha k_2} \triangle^{k_2} (-\gamma)^{k_3} I_t^{\alpha k_3} \nabla^{k_3} \\ &= \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} (-a)^{k_1} b^{k_2} \gamma^{k_3} I_t^{\alpha k-\beta k_1} \triangle^{k_2} \nabla^{k_3} \end{aligned}$$

in the space $S$. Indeed, for any $\phi(t,\mathbf{x}) \in S$, we have

$$V\phi(t,\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} (-a)^{k_1} b^{k_2} \gamma^{k_3} I_t^{\alpha k-\beta k_1} \triangle^{k_2} \nabla^{k_3} \phi(t,\mathbf{x}),$$

and

$$|V\phi(t,\mathbf{x})| \leq \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} |a|^{k_1} |b|^{k_2} |\gamma|^{k_3} I_t^{\alpha k-\beta k_1} |\triangle^{k_2} \nabla^{k_3} \phi(t,\mathbf{x})|.$$

Clearly,

$$\begin{aligned} \triangle^{k_2} \nabla^{k_3} \phi(t,\mathbf{x}) &= \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^{k_2} \left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}\right)^{k_3} \phi(t,\mathbf{x}) \\ &= \sum_{j_1+\cdots+j_n=k_2} \binom{k_2}{j_1,\cdots,j_n} \sum_{i_1+\cdots+i_n=k_3} \binom{k_3}{i_1,\cdots,i_n} \\ &\quad \cdot \left(\frac{\partial}{\partial x_1}\right)^{2j_1+i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{2j_n+i_n} \phi(t,\mathbf{x}), \end{aligned}$$

which implies that

$$\begin{aligned} |\triangle^{k_2} \nabla^{k_3} \phi(t,\mathbf{x})| &\leq \theta(t,\mathbf{x}) M_\phi^{2k_2+k_3} \sum_{j_1+\cdots+j_n=k_2} \binom{k_2}{j_1,\cdots,j_n} \sum_{i_1+\cdots+i_n=k_3} \binom{k_3}{i_1,\cdots,i_n} \\ &= \theta(t,\mathbf{x}) \left(M_\phi^2 n\right)^{k_2} (M_\phi n)^{k_3}, \end{aligned}$$

using

$$\sum_{j_1+\cdots+j_n=k} \binom{k}{j_1,\cdots,j_n} = n^k.$$

Then, for every fixed $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$, we have

$$\begin{aligned} |V\phi(t,\mathbf{x})| &\leq \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} |a|^{k_1} |b|^{k_2} |\gamma|^{k_3} \left(M_\phi^2 n\right)^{k_2} (M_\phi n)^{k_3} I_t^{\alpha k-\beta k_1} \theta(t,\mathbf{x}) \\ &\leq \max_{0\leq\tau\leq t} \theta(\tau,\mathbf{x}) \cdot \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1,k_2,k_3} \frac{(|a| t^{\alpha-\beta})^{k_1} (|b| M_\phi^2 n t^{\alpha})^{k_2} (|\gamma| M_\phi n t^{\alpha})^{k_3}}{\Gamma((\alpha-\beta)k_1 + \alpha k_2 + \alpha k_3 + 1)} \\ &= \max_{0\leq\tau\leq t} \theta(\tau,\mathbf{x}) \cdot E_{(\alpha-\beta,\alpha,\alpha),\,1}\left(|a| t^{\alpha-\beta}, |b| M_\phi^2 n t^{\alpha}, |\gamma| M_\phi n t^{\alpha}\right) < +\infty. \end{aligned}$$

In summary, the operator $V$ is well-defined over the space $S$. We further prove that $V$ is a unique inverse operator as

$$\begin{aligned}V\big(1 + a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big) &= \big(1 + a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big)V \\ &= 1 \ \ \text{(identity operator)}.\end{aligned}$$

In fact,

$$\begin{aligned}V\big(1 + a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big) &= 1 + \sum_{k=1}^{\infty}(-1)^k\big(a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big)^k \\ &\quad + \sum_{k=0}^{\infty}(-1)^k\big(a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big)^{k+1} \\ &= 1 + \sum_{k=0}^{\infty}(-1)^{k+1}\big(a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big)^{k+1} \\ &\quad + \sum_{k=0}^{\infty}(-1)^k\big(a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big)^{k+1} \\ &= 1.\end{aligned}$$

Similarly, $\big(1 + a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big)V = 1.$ Assuming $V'$ is another inverse operator, we have

$$\big(1 + a\, I_t^{\alpha-\beta} - b\, I_t^{\alpha}\triangle - \gamma\, I_t^{\alpha}\nabla\big)V' = 1.$$

Applying $V$ to both sides, we obtain $V = V'$. From Equation (8), we obtain

$$\begin{aligned}W(t,\mathbf{x}) &= \sum_{k=0}^{\infty}\sum_{k_1+k_2+k_3=k}\binom{k}{k_1,k_2,k_3}(-a)^{k_1}b^{k_2}\gamma^{k_3}I_t^{\alpha k-\beta k_1}\triangle^{k_2}\nabla^{k_3} \\ &\quad\cdot\left(I_t^{\alpha}\phi(t,\mathbf{x}) + \psi(\mathbf{x}) + \frac{a\,\psi(\mathbf{x})\,t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) \\ &= \sum_{k=0}^{\infty}\sum_{k_1+k_2+k_3=k}\binom{k}{k_1,k_2,k_3}(-a)^{k_1}b^{k_2}\gamma^{k_3}I_t^{\alpha k-\beta k_1+\alpha}\triangle^{k_2}\nabla^{k_3}\phi(t,\mathbf{x}) \\ &\quad + \sum_{k=0}^{\infty}\sum_{k_1+k_2+k_3=k}\binom{k}{k_1,k_2,k_3}\frac{(-at^{\alpha-\beta})^{k_1}(bt^{\alpha})^{k_2}(\gamma t^{\alpha})^{k_3}}{\Gamma((\alpha-\beta)k_1+\alpha k_2+\alpha k_3+1)}\triangle^{k_2}\nabla^{k_3}\psi(\mathbf{x}) \\ &\quad + at^{\alpha-\beta}\sum_{k=0}^{\infty}\sum_{k_1+k_2+k_3=k}\binom{k}{k_1,k_2,k_3} \\ &\qquad\cdot\frac{(-at^{\alpha-\beta})^{k_1}(bt^{\alpha})^{k_2}(\gamma t^{\alpha})^{k_3}}{\Gamma((\alpha-\beta)k_1+\alpha k_2+\alpha k_3+\alpha-\beta+1)}\triangle^{k_2}\nabla^{k_3}\psi(\mathbf{x}).\end{aligned}$$

The uniqueness of solutions follows from the uniqueness of the inverse operator $V$. This completes the proof. □

**Remark 1.** *When applied to the diffusion Equation (1) in dimensions greater than one, the fractional time derivatives induce divergent behavior at the origin. This has profound physical implications, reflecting non-local memory effects, anomalous transport, and power-law relaxation processes.*

## 3. The Solution to the Time-Fractional Non-Homogenous Convection Equation

Clearly, if $a = b = 0$, then the fractional convection equation:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} W(t,\mathbf{x}) = \gamma \nabla W(t,\mathbf{x}) + \phi(t,\mathbf{x}), \\ W(0,\mathbf{x}) = \psi(\mathbf{x}), \end{cases} \tag{9}$$

has a unique solution for $0 < \alpha \leq 1$ from Theorem 2

$$W(t,\mathbf{x}) = \sum_{k=0}^{\infty} \gamma^k I_t^{\alpha k+\alpha} \nabla^k \phi(t,\mathbf{x}) + \sum_{k=0}^{\infty} \frac{(\gamma t^\alpha)^k}{\Gamma(\alpha k+1)} \nabla^k \psi(\mathbf{x}). \tag{10}$$

In the following, we show that solution (10) can be written as the following convolution solution for $\mathbf{x} \in \mathbb{R}^n$ (the AI model DeepSeek-$R_1$ finds this integral solution):

$$W(t,\mathbf{x}) = \int_{\mathbb{R}^n} \psi(\mathbf{y}) G_1(\mathbf{x}-\mathbf{y},t) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \phi(\tau,\mathbf{y}) G_2(\mathbf{x}-\mathbf{y},t-\tau) d\mathbf{y} d\tau, \tag{11}$$

where $G_1(\mathbf{x},t)$ and $G_2(\mathbf{x},t)$ are Green's functions, given by

$$G_1(\mathbf{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha\left(\mathrm{i}\gamma|\boldsymbol{\xi}|t^\alpha\right) d\boldsymbol{\xi},$$

and

$$G_2(\mathbf{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} t^{\alpha-1} E_{\alpha,\alpha}(\gamma \mathrm{i}|\boldsymbol{\xi}|t^\alpha) d\boldsymbol{\xi},$$

respectively.

We begin defining the $n$-dimensional Fourier transform as

$$\mathcal{F}\{\psi\}(\boldsymbol{\xi}) = \tilde{\psi}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} \psi(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} d\mathbf{x}, \quad \boldsymbol{\xi} = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n,$$

and the inverse Fourier transform

$$\psi(\mathbf{x}) = \mathcal{F}^{-1}\{\tilde{\psi}\}(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} d\boldsymbol{\xi}.$$

where $\langle \boldsymbol{\xi}, \mathbf{x}\rangle := \xi_1 x_1 + \cdots + \xi_n x_n$. In particular, we write $|\boldsymbol{\xi}| = \xi_1 + \cdots + \xi_n$. Then, we have $\mathcal{F}\{\nabla\psi\}(\boldsymbol{\xi}) = \mathrm{i}|\boldsymbol{\xi}|\tilde{\psi}(\boldsymbol{\xi})$, and thus,

$$\nabla^j \psi(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\mathrm{i}|\boldsymbol{\xi}|)^j \tilde{\psi}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} d\boldsymbol{\xi}.$$

So, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(\gamma t^\alpha)^j}{\Gamma(\alpha j+1)} \nabla^j \psi(\mathbf{x}) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} \left[ \sum_{j=0}^{\infty} \frac{(\mathrm{i}\gamma|\boldsymbol{\xi}|t^\alpha)^j}{\Gamma(\alpha j+1)} \right] d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha\left(\mathrm{i}\gamma|\boldsymbol{\xi}|t^\alpha\right) d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \psi(\mathbf{y}) \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi},\mathbf{y}\rangle} d\mathbf{y} \right] \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha\left(\mathrm{i}\gamma|\boldsymbol{\xi}|t^\alpha\right) d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(\mathbf{y}) \left[ \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}-\mathbf{y}\rangle} E_\alpha\left(\mathrm{i}\gamma|\boldsymbol{\xi}|t^\alpha\right) d\boldsymbol{\xi} \right] d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \psi(\mathbf{y}) G_1(\mathbf{x}-\mathbf{y},t) d\mathbf{y}. \end{aligned} \tag{12}$$

On the other hand, we have

$$\sum_{j=0}^{\infty}\gamma^j I_t^{\alpha j+\alpha}\nabla^j\phi(t,\mathbf{x}) = \frac{1}{(2\pi)^n}\sum_{j=0}^{\infty}\gamma^j I_t^{\alpha j+\alpha}\int_{\mathbb{R}^n}(\mathrm{i}|\boldsymbol{\xi}|)^j \mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\mathbf{x}\rangle}\tilde{\phi}(t,\boldsymbol{\xi})d\boldsymbol{\xi}$$
$$= \frac{1}{(2\pi)^n}\sum_{j=0}^{\infty}\gamma^j I_t^{\alpha j+\alpha}\int_{\mathbb{R}^n}(\mathrm{i}|\boldsymbol{\xi}|)^j \mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\mathbf{x}\rangle}\int_{\mathbb{R}^n}\phi(t,\mathbf{y})\mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\xi},\mathbf{y}\rangle}d\mathbf{y}d\boldsymbol{\xi}$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\mathbf{x}\rangle}\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\xi},\mathbf{y}\rangle}\left[\sum_{j=0}^{\infty}(\gamma\mathrm{i}|\boldsymbol{\xi}|)^j I_t^{\alpha j+\alpha}\phi(t,\mathbf{y})\right]d\mathbf{y}d\boldsymbol{\xi}.$$

It is not difficult to see that

$$\sum_{j=0}^{\infty}(\gamma\mathrm{i}|\boldsymbol{\xi}|)^j I_t^{\alpha j+\alpha}\phi(t,\mathbf{y}) = \sum_{j=0}^{\infty}(\gamma\mathrm{i}|\boldsymbol{\xi}|)^j\int_0^t\frac{(t-\tau)^{\alpha j+\alpha-1}}{\Gamma(\alpha j+\alpha)}\phi(\tau,\mathbf{y})d\tau$$
$$= \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\gamma\mathrm{i}|\boldsymbol{\xi}|(t-\tau)^{\alpha})\phi(\tau,\mathbf{y})d\tau.$$

So, we have

$$\sum_{j=0}^{\infty}\gamma^j I_t^{\alpha j+\alpha}\nabla^j\phi(t,\mathbf{x})$$
$$= \frac{1}{(2\pi)^n}\int_0^t\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\mathbf{x}-\mathbf{y}\rangle}E_{\alpha,\alpha}(\gamma\mathrm{i}|\boldsymbol{\xi}|(t-\tau)^{\alpha})d\boldsymbol{\xi}\right](t-\tau)^{\alpha-1}\phi(\tau,\mathbf{y})d\mathbf{y}d\tau$$
$$= \int_0^t\int_{\mathbb{R}^n}G_2(\mathbf{x}-\mathbf{y},t-\tau)\phi(\tau,\mathbf{y})d\mathbf{y}d\tau. \tag{13}$$

Substituting (12) and (13) into (10) results in the desired convolution expression (11).

Furthermore, if $\alpha = 1$, $\phi(t,x) = 0$ and $n = 1$, then the equation:

$$\begin{cases}\dfrac{\partial}{\partial t}W(t,x) - \gamma\dfrac{\partial}{\partial x}W(t,x) = 0,\\ W(0,x) = \psi(x),\end{cases} \tag{14}$$

has a unique solution by noting that $\psi \in S$

$$W(t,x) = \sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\left(\frac{\partial}{\partial x}\right)^k\psi(x) = \psi(x+\gamma t),$$

which represents a wave travelling to the right if $\gamma < 0$ or left if $\gamma > 0$.

On the other hand, if $\alpha = 1$ and $n = 1$, then we obtain Equation (2) with a unique solution

$$W(t,x) = \sum_{k=0}^{\infty}\frac{(\gamma t)^k}{k!}\left(\frac{\partial}{\partial x}\right)^k\psi(x) + \sum_{k=0}^{\infty}\gamma^k I_t^{k+1}\left(\frac{\partial}{\partial x}\right)^k\phi(t,x)$$
$$= \psi(x+\gamma t) + \sum_{k=0}^{\infty}\frac{\gamma^k}{k!}\int_0^t (t-\tau)^k\left(\frac{\partial}{\partial x}\right)^k\phi(\tau,x)d\tau$$
$$= \psi(x+\gamma t) + \int_0^t\phi(\tau,x+\gamma(t-\tau))d\tau.$$

**Example 1.** *The following time-fractional convection equation:*

$$\begin{cases}\dfrac{{}_c\partial^{\alpha}}{\partial t^{\alpha}}W(t,\mathbf{x}) = \nabla W(t,\mathbf{x}) + t|\mathbf{x}|^2,\\ W(0,\mathbf{x}) = x_n^m, \quad m\in\mathbb{N},\end{cases} \tag{15}$$

*has a unique solution for* $0 < \alpha \leq 1$

$$\begin{aligned} W(t,\mathbf{x}) &= \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha} \nabla^k \phi(t,\mathbf{x}) + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \nabla^k \psi(\mathbf{x}) \\ &= \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha} t \nabla^k |\mathbf{x}|^2 + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \nabla^k x_n^m. \end{aligned}$$

*Clearly, we have*

$$\nabla^k |\mathbf{x}|^2 = \begin{cases} 1, & k=0, \\ 2n|\mathbf{x}|, & k=1, \\ 2n^2, & k=2, \\ 0, & k \geq 3 \end{cases}$$

*and*

$$\nabla^k x_n^m = \begin{cases} (-1)^k (-m)_k \, x_n^{m-k}, & 0 \leq k \leq m, \\ 0, & k \geq m+1, \end{cases}$$

*where* $(\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1)$. *Thus,*

$$W(t,\mathbf{x}) = \frac{t^{\alpha+1}|\mathbf{x}|^2}{\Gamma(\alpha+2)} + \frac{2nt^{2\alpha+1}|\mathbf{x}|}{\Gamma(2\alpha+2)} + \frac{2n^2 t^{3\alpha+1}}{\Gamma(3\alpha+2)} + x_n^m \sum_{k=0}^{m} \frac{(-m)_k}{\Gamma(\alpha k+1)} \left(-\frac{t^\alpha}{x_n}\right)^k.$$

We should note that it would be complicated to use the following integral formula to find the solution:

$$W(t,\mathbf{x}) = \int_{\mathbb{R}^n} \psi(\mathbf{y}) G_1(\mathbf{x}-\mathbf{y},t) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \phi(\tau,\mathbf{y}) G_2(\mathbf{x}-\mathbf{y},t-\tau) d\mathbf{y} d\tau,$$

and our method is much easier and faster.

**Remark 2.** *(a) It would be interesting to see how DeepSeek-R1 finds the solution* (11)*. When the problem* (9) *is entered in LaTeX code form, DeepSeek-R1 first converts the fractional partial differential equation into the partial differential equation*

$$(s^\alpha - \gamma\nabla)\widetilde{W}(s,\mathbf{x}) = s^{\alpha-1}\psi(\mathbf{x}) + \widetilde{\phi}(s,\mathbf{x}) \tag{16}$$

*by performing the Laplace transform* $\mathcal{L}$. *Here,* $\widetilde{W}(s,\mathbf{x}) = \mathcal{L}[W(t,\mathbf{x})](s)$ *and* $\widetilde{\phi}(s,\mathbf{x}) = \mathcal{L}[\phi(t,\mathbf{x})]$. *Then DeepSeek-R1 points out that Equation* (16) *can be solved by using the Fourier transform method. After applying the inverse Laplace transform and inverse Fourier transform, it obtains the solution* (11)*. The authors also employ DeepSeek-R1 to find series solutions of* (9)*, but it* fails to reach *our expression* (10)*. The reason is that DeepSeek AI can generally be used to solve many fractional partial differential equations (FPDEs) analytically by leveraging known techniques such as Laplace and Fourier transforms, separation of variables, Green's functions for fundamental solutions, eigenfunction expansions, and series expansions with re-summation. However, our series solution method differs from the approaches mentioned above.*

*(b) The experience of using DeepSeek in this specific case has been notably effective. It not only presented integral convolution solutions but also offered detailed steps mentioned above that accelerated our understanding of the existing theoretical framework. By streamlining the literature review process, DeepSeek allowed us to focus more deeply on the key materials of the problem, ensuring that our work builds directly upon them.*

## 4. The Solution to the Time-Fractional Non-Homogenous Diffusion Equation

Clearly, if $a = \gamma = 0$, then the fractional non-homogenous diffusion equation:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} W(t,\mathbf{x}) = b \triangle W(t,\mathbf{x}) + \phi(t,\mathbf{x}), \\ W(0,\mathbf{x}) = \psi(\mathbf{x}), \end{cases} \tag{17}$$

has a unique solution for $0 < \alpha \le 1$ from Theorem 2

$$W(t,\mathbf{x}) = \sum_{k=0}^{\infty} b^k I_t^{\alpha k+\alpha} \triangle^k \phi(t,\mathbf{x}) + \sum_{k=0}^{\infty} \frac{(bt^\alpha)^k}{\Gamma(\alpha k+1)} \triangle^k \psi(\mathbf{x}). \tag{18}$$

In the following, we prove that solution (18) can be written as the following convolution solution for $\mathbf{x} \in \mathbb{R}^n$ (the AI model DeepSeek-$R_1$ claims this integral solution):

$$W(t,\mathbf{x}) = \int_{\mathbb{R}^n} \psi(\mathbf{y}) G_1(\mathbf{x}-\mathbf{y}, t) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \phi(\tau,\mathbf{y}) G_2(\mathbf{x}-\mathbf{y}, t-\tau) d\mathbf{y} d\tau, \tag{19}$$

where $G_1(\mathbf{x},t)$ and $G_2(\mathbf{x},t)$ are Green's functions, given by

$$G_1(\mathbf{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha(-b|\boldsymbol{\xi}|^2 t^\alpha) d\boldsymbol{\xi},$$

and

$$G_2(\mathbf{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} t^{\alpha-1} E_{\alpha,\alpha}(-b|\boldsymbol{\xi}|^2 t^\alpha) d\boldsymbol{\xi},$$

respectively.

In fact, from $\mathcal{F}\{\triangle\psi\}(\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^2 \tilde{\psi}(\boldsymbol{\xi})$, we have

$$\triangle^j \psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-|\boldsymbol{\xi}|^2)^j \tilde{\psi}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} d\boldsymbol{\xi}.$$

Then,

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(bt^\alpha)^j}{\Gamma(\alpha j+1)} \triangle^j \psi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} \left[ \sum_{j=0}^{\infty} \frac{(-b|\boldsymbol{\xi}|^2 t^\alpha)^j}{\Gamma(\alpha j+1)} \right] d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha(-b|\boldsymbol{\xi}|^2 t^\alpha) d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \psi(\mathbf{y}) \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi},\mathbf{y}\rangle} d\mathbf{y} \right] \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha(-b|\boldsymbol{\xi}|^2 t^\alpha) d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(\mathbf{y}) \left[ \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}-\mathbf{y}\rangle} E_\alpha(-b|\boldsymbol{\xi}|^2 t^\alpha) d\boldsymbol{\xi} \right] d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \psi(\mathbf{y}) G_1(\mathbf{x}-\mathbf{y}, t) d\mathbf{y} \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
&\sum_{j=0}^{\infty} b^j I_t^{\alpha j+\alpha} \triangle^j \phi(t,\mathbf{x}) \\
&= \frac{1}{(2\pi)^n} \sum_{j=0}^{\infty} b^j I_t^{\alpha j+\alpha} \int_{\mathbb{R}^n} (-|\boldsymbol{\xi}|^2)^j \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} \int_{\mathbb{R}^n} \phi(t,\mathbf{y}) \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi},\mathbf{y}\rangle} d\mathbf{y} d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi},\mathbf{y}\rangle} \left[ \sum_{j=0}^{\infty} (-b|\boldsymbol{\xi}|^2)^j I_t^{\alpha j+\alpha} \phi(t,\mathbf{y}) \right] d\mathbf{y} d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}\rangle} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi},\mathbf{y}\rangle} \left[ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-b|\boldsymbol{\xi}|^2 (t-\tau)^{\alpha}) \phi(\tau,\mathbf{y}) d\tau \right] d\mathbf{y} d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^n} \int_0^t \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi},\mathbf{x}-\mathbf{y}\rangle} E_{\alpha,\alpha}(-b|\boldsymbol{\xi}|^2 (t-\tau)^{\alpha}) d\boldsymbol{\xi} \right] (t-\tau)^{\alpha-1} \phi(\tau,\mathbf{y}) d\mathbf{y} d\tau \\
&= \int_0^t \int_{\mathbb{R}^n} G_2(\mathbf{x}-\mathbf{y}, t-\tau) \phi(\tau,\mathbf{y}) d\mathbf{y} d\tau. \qquad (21)
\end{aligned}
$$

The result (19) follows immediately by substituting (20) and (21) into (18).

**Remark 3.** *DeepSeek-R1 employs Fourier transform and Laplace transform to solve the problem (17) and obtain (19), in the same way as when solving the problem (9). However, it is unable to derive our series solution.*

**Example 2.** *The following time-fractional diffusion equation:*

$$
\begin{cases}
\frac{{}_c\partial^{\alpha}}{\partial t^{\alpha}} W(t,\mathbf{x}) = 2 \triangle W(t,\mathbf{x}) + t^2 |\mathbf{x}|^4, \\
W(0,\mathbf{x}) = \sin|\mathbf{x}|,
\end{cases} \qquad (22)
$$

*has a unique solution for* $0 < \alpha \leq 1$

$$
\begin{aligned}
W(t,\mathbf{x}) &= \sum_{k=0}^{\infty} b^k I_t^{\alpha k+\alpha} \triangle^k \phi(t,\mathbf{x}) + \sum_{k=0}^{\infty} \frac{(bt^{\alpha})^k}{\Gamma(\alpha k+1)} \triangle^k \psi(\mathbf{x}) \\
&= \sum_{k=0}^{\infty} 2^k (I_t^{\alpha k+\alpha} t^2)(\triangle^k |\mathbf{x}|^4) + \sum_{k=0}^{\infty} \frac{(2t^{\alpha})^k}{\Gamma(\alpha k+1)} \triangle^k \sin|\mathbf{x}|.
\end{aligned}
$$

*Clearly,* $\sin|\mathbf{x}| \in S$ *and* $\triangle^k \sin|\mathbf{x}| = (-n)^k \sin|\mathbf{x}|$. *Through calculation, we have*

$$
\triangle |\mathbf{x}|^4 = 12n|\mathbf{x}|^2,
$$

*which implies that*

$$
\triangle^2 |\mathbf{x}|^4 = 12n \triangle |\mathbf{x}|^2 = 24n^2.
$$

*Therefore,*

$$
\begin{aligned}
W(t,\mathbf{x}) &= |\mathbf{x}|^4 I_t^{\alpha} t^2 + 24n|\mathbf{x}|^2 I_t^{2\alpha} t^2 + 96n^2 I_t^{3\alpha} t^2 + \sum_{k=0}^{\infty} \frac{(-2nt^{\alpha})^k}{\Gamma(\alpha k+1)} \sin|\mathbf{x}| \\
&= \frac{2t^{\alpha+2}|\mathbf{x}|^4}{\Gamma(\alpha+3)} + \frac{48nt^{2\alpha+2}|\mathbf{x}|^2}{\Gamma(2\alpha+3)} + \frac{192n^2 t^{3\alpha+2}}{\Gamma(3\alpha+3)} + E_{\alpha}(-2nt^{\alpha}) \sin|\mathbf{x}|,
\end{aligned}
$$

*by noting that* $t^2|\mathbf{x}|^4 \in S$. *This approach to finding the solution is much easier and simpler than directly using the following formula:*

$$
W(t,\mathbf{x}) = \int_{\mathbb{R}^n} \psi(\mathbf{y}) G_1(\mathbf{x}-\mathbf{y}, t) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \phi(\tau,\mathbf{y}) G_2(\mathbf{x}-\mathbf{y}, t-\tau) d\mathbf{y} d\tau.
$$

## 5. The Fractional Convection-Diffusion Equation

It follows from $a = 0$ that Equation (1) becomes the following fractional convection–diffusion equation:

$$\begin{cases} {}_c\dfrac{\partial^\alpha}{\partial t^\alpha} W(t,\mathbf{x}) = b \triangle W(t,\mathbf{x}) + \gamma \nabla W(t,\mathbf{x}) + \phi(t,\mathbf{x}), \\ W(0,x) = \psi(\mathbf{x}), \end{cases} \tag{23}$$

with a unique solution from Theorem 2:

$$\begin{aligned} W(t,\mathbf{x}) = & \sum_{k=0}^{\infty} I_t^{\alpha k+\alpha} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} b^{k_1}\gamma^{k_2} \triangle^{k_1} \nabla^{k_2} \phi(t,\mathbf{x}) \\ & + \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} b^{k_1}\gamma^{k_2} \triangle^{k_1} \nabla^{k_2} \psi(\mathbf{x}). \end{aligned} \tag{24}$$

We will show that solution (23) can be written as the following convolution solution for $\mathbf{x} \in \mathbb{R}^n$ (the AI model DeepSeek-$R_1$ also claims this integral solution):

$$W(t,\mathbf{x}) = \int_{\mathbb{R}^n} \psi(\mathbf{y}) G_1(\mathbf{x}-\mathbf{y},t) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \phi(\tau,\mathbf{y}) G_2(\mathbf{x}-\mathbf{y},t-\tau) d\mathbf{y} d\tau,$$

where $G_1(\mathbf{x},t)$ and $G_2(\mathbf{x},t)$ are Green's functions, given by

$$G_1(\mathbf{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha(-(b|\boldsymbol{\xi}|^2 - i\gamma|\boldsymbol{\xi}|)t^\alpha) d\boldsymbol{\xi}$$

and

$$G_2(\mathbf{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} t^{\alpha-1} E_{\alpha,\alpha}(-(b|\boldsymbol{\xi}|^2 - \gamma i|\boldsymbol{\xi}|)t^\alpha) d\boldsymbol{\xi},$$

respectively.

Since $\mathcal{F}\{\triangle^j \psi\}(\boldsymbol{\xi}) = (-|\boldsymbol{\xi}|^2)^j \tilde{\psi}(\boldsymbol{\xi})$ and $\mathcal{F}\{\nabla^j \psi\}(\boldsymbol{\xi}) = (i|\boldsymbol{\xi}|)^j \tilde{\psi}(\boldsymbol{\xi})$, we have

$$\begin{aligned} \mathcal{F}\{\triangle^{j_1} \nabla^{j_2} \psi\}(\boldsymbol{\xi}) &= (-|\boldsymbol{\xi}|^2)^{j_1} \mathcal{F}\{\nabla^{j_2}\psi\}(\boldsymbol{\xi}) \\ &= (-1)^{j_1} |\boldsymbol{\xi}|^{2j_1} (i|\boldsymbol{\xi}|)^{j_2} \tilde{\psi}(\boldsymbol{\xi}), \end{aligned}$$

and, thus,

$$\triangle^{j_1} \nabla^{j_2} \psi(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-1)^{j_1} |\boldsymbol{\xi}|^{2j_1} (i|\boldsymbol{\xi}|)^{j_2} \tilde{\psi}(\boldsymbol{\xi}) e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} d\boldsymbol{\xi}.$$

Then,

$$\begin{aligned} & \sum_{j=0}^{\infty} \frac{t^{\alpha j}}{\Gamma(\alpha j+1)} \sum_{j_1+j_2=j} \binom{j}{j_1,j_2} b^{j_1} \gamma^{j_2} \triangle^{j_1} \nabla^{j_2} \psi(\mathbf{x}) \\ & = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\boldsymbol{\xi}) e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} \left[ \sum_{j=0}^{\infty} \frac{t^{\alpha j}}{\Gamma(\alpha j+1)} \sum_{j_1+j_2=j} \binom{j}{j_1,j_2} (-b|\boldsymbol{\xi}|^2)^{j_1} (i\gamma|\boldsymbol{\xi}|)^{j_2} \right] d\boldsymbol{\xi} \\ & = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\boldsymbol{\xi}) e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} \left[ \sum_{j=0}^{\infty} \frac{(-(b|\boldsymbol{\xi}|^2 - i\gamma|\boldsymbol{\xi}|)t^\alpha)^j}{\Gamma(\alpha j+1)} \right] d\boldsymbol{\xi} \\ & = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \psi(\mathbf{y}) e^{-i\langle \boldsymbol{\xi},\mathbf{y}\rangle} d\mathbf{y} \right] e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} E_\alpha(-(b|\boldsymbol{\xi}|^2 - i\gamma|\boldsymbol{\xi}|)t^\alpha) d\boldsymbol{\xi} \\ & = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(\mathbf{y}) \left[ \int_{\mathbb{R}^n} e^{i\langle \boldsymbol{\xi},\mathbf{x}-\mathbf{y}\rangle} E_\alpha(-(b|\boldsymbol{\xi}|^2 - i\gamma|\boldsymbol{\xi}|)t^\alpha) d\boldsymbol{\xi} \right] e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} d\mathbf{y} \\ & = \int_{\mathbb{R}^n} \psi(y) G_1(\mathbf{x}-\mathbf{y},t) d\mathbf{y}. \end{aligned}$$

and

$$
\begin{aligned}
&\sum_{j=0}^{\infty} I_t^{\alpha j+\alpha} \sum_{j_1+j_2=j} \binom{j}{j_1, j_2} b^{j_1} \gamma^{j_2} \triangle^{j_1} \nabla^{j_2} \phi(t, \mathbf{x}) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \xi, \mathbf{x} \rangle} \sum_{j=0}^{\infty} I_t^{\alpha j+\alpha} \sum_{j_1+j_2=j} \binom{j}{j_1, j_2} (-b|\boldsymbol{\xi}|^2)^{j_1} (\gamma \mathrm{i} |\boldsymbol{\xi}|)^{j_2} \tilde{\phi}(t, \boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \xi, \mathbf{x} \rangle} \sum_{j=0}^{\infty} (-b|\boldsymbol{\xi}|^2 + \gamma \mathrm{i} |\boldsymbol{\xi}|)^j I_t^{\alpha j+\alpha} \tilde{\phi}(t, \boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^n} \int_0^t (t-\tau)^{\alpha-1} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \xi, \mathbf{x} \rangle} E_{\alpha,\alpha}(-(b|\boldsymbol{\xi}|^2 - \gamma \mathrm{i} |\boldsymbol{\xi}|)(t-\tau)^\alpha) \tilde{\phi}(\tau, \boldsymbol{\xi}) d\boldsymbol{\xi} d\tau \\
&= \frac{1}{(2\pi)^n} \int_0^t (t-\tau)^{\alpha-1} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \xi, \mathbf{x} \rangle} E_{\alpha,\alpha}(-(b|\boldsymbol{\xi}|^2 - \gamma \mathrm{i} |\boldsymbol{\xi}|)(t-\tau)^\alpha) \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \xi, \mathbf{y} \rangle} \phi(\tau, \mathbf{y}) d\mathbf{y} d\boldsymbol{\xi} d\tau \\
&= \frac{1}{(2\pi)^n} \int_0^t (t-\tau)^{\alpha-1} \int_{\mathbb{R}^n} \phi(\tau, \mathbf{y}) \left[ \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \xi, \mathbf{x}-\mathbf{y} \rangle} E_{\alpha,\alpha}(-(b|\boldsymbol{\xi}|^2 - \gamma \mathrm{i} |\boldsymbol{\xi}|)(t-\tau)^\alpha) d\boldsymbol{\xi} \right] d\mathbf{y} d\tau \\
&= \int_0^t \int_{\mathbb{R}^n} \phi(\tau, \mathbf{y}) G_2(\mathbf{x}-\mathbf{y}, t-\tau) d\mathbf{y} d\tau.
\end{aligned}
$$

**Example 3.** *The following fractional convection–diffusion equation:*

$$
\begin{cases}
\frac{{}_c\partial^\alpha}{\partial t^\alpha} W(t, \mathbf{x}) = \triangle W(t, \mathbf{x}) + \nabla W(t, \mathbf{x}) + \sin(t^2|\mathbf{x}|), \\
W(0, \mathbf{x}) = 1,
\end{cases}
\tag{25}
$$

*has a unique solution for* $0 < \alpha \leq 1$

$$
\begin{aligned}
W(t, \mathbf{x}) = &\sum_{k=0}^{\infty} I_t^{\alpha k+\alpha} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \triangle^{k_1} \nabla^{k_2} \sin(t^2|\mathbf{x}|) \\
&+ \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \triangle^{k_1} \nabla^{k_2} 1.
\end{aligned}
$$

*Clearly,*

$$
\begin{aligned}
\nabla \sin(t^2|\mathbf{x}|) &= n t^2 \sin\Big(t^2|\mathbf{x}| + \frac{\pi}{2}\Big), \\
&\vdots \\
\nabla^{k_2} \sin(t^2|\mathbf{x}|) &= (n t^2)^{k_2} \sin\left(t^2|\mathbf{x}| + \frac{k_2 \pi}{2}\right), \\
\triangle \nabla^{k_2} \sin(t^2|\mathbf{x}|) &= (n t^2)^{k_2} n t^4 \sin\left(t^2|\mathbf{x}| + \frac{k_2 \pi}{2} + \pi\right), \\
&\vdots \\
\triangle^{k_1} \nabla^{k_2} \sin(t^2|\mathbf{x}|) &= (n t^2)^{k_2} (n t^4)^{k_1} \sin\left(t^2|\mathbf{x}| + \frac{k_2 \pi}{2} + k_1 \pi\right) \\
&= n^{k_1+k_2} t^{4k_1+2k_2} (-1)^{k_1} \sin\left(t^2|\mathbf{x}| + \frac{k_2 \pi}{2}\right).
\end{aligned}
$$

*Hence,*

$$W(t,\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} n^{k_1+k_2} I_t^{\alpha k+\alpha} t^{4k_1+2k_2} (-1)^{k_1} \sin\left(t^2|\mathbf{x}| + \frac{k_2\pi}{2}\right)$$
$$+ \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} \triangle^{k_1} \nabla^{k_2} 1$$
$$= 1 + \sum_{k=0}^{\infty} n^k \sum_{k_1+k_2=k} (-1)^{k_1} \binom{k}{k_1,k_2} I_t^{\alpha k+\alpha} t^{4k_1+2k_2} \sin\left(t^2|\mathbf{x}| + \frac{k_2\pi}{2}\right).$$

We must add that it would be complicated to use the above convolution formula to find the solution.

**Remark 4.** *(a) The highly complex integral solutions using Green's functions for Examples 7 and 8 can be obtained with DeepSeek-R1. However, it is unable to find the simple series solutions that we derived. The reason for this was explained earlier in Remark 5 (a).*

*(b) We also note that handling Equation (1) with a variable coefficient, such as $a = a(t,\mathbf{x})$, or certain boundary conditions, would be challenging using our current inverse operator method, as it appears infeasible to explicitly construct an inverse operator.*

## 6. Conclusions

Using the inverse operator approach and the multivariate Mittag–Leffler function, we derived an analytic solution to the time-fractional convection–diffusion Equation (1) and showed that our formula is consistent with the integral convolution solution using the Fourier transforms and Taylor's expansions of the Mittag–Leffler functions. In addition, a few examples were presented to demonstrate the effectiveness of our method in solving certain fractional convection–diffusion equations. Our inverse approach also serves as a powerful tool for investigating a wide range of other partial differential equations with variable coefficients. As a direction for future research, it is worth studying the following time-fractional Klein–Gordon equation for $0 < \alpha \leq 1$:

$$\begin{cases} {}_c\dfrac{\partial^{2\alpha}}{\partial t^{2\alpha}} W(t,\mathbf{x}) - \triangle W(t,\mathbf{x}) + m^{2\alpha} W(t,\mathbf{x}) = 0, \\ W(0,\mathbf{x}) = \psi_1(\mathbf{x}), \quad W_t'(0,\mathbf{x}) = \psi_2(\mathbf{x}), \quad (t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n, \end{cases}$$

where $m$ is a constant (mass parameter).

The time-fractional Klein–Gordon equation is a useful mathematical tool that extends the classical Klein–Gordon equation by incorporating fractional derivatives in time. This generalization allows the equation to model phenomena with memory effects, non-local behavior, and anomalous dynamics.

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