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Studies on Fractional Differential Equations With Functional Boundary Condition by Inverse Operators

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ABSTRACT

Fractional differential equations (FDEs) generalize classical integer-order calculus to noninteger orders, enabling the modeling of complex phenomena that classical equations cannot fully capture. Their study has become essential across science, engineering, and mathematics due to their unique ability to describe systems with nonlocal interactions. In this paper, we study the uniqueness, existence, and stability for a new nonlinear FDE with functional boundary condition (which describes nonlocal properties) based on several well-known fixed-point theorems, the two-parameter Mittag–Leffler function, and an implicit integral equation obtained from inverse operators. Several examples are presented to demonstrate applications of our key theorems. Furthermore, we also indicate that the method used can deal with PDEs, with various initial or boundary conditions.

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1 | Introduction and Preliminaries

The Riemann–Liouville fractional integral I^β of order $\beta \in R^+$ is defined for the function $\zeta(x)$ as

$$(I^\beta \zeta)(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-\tau)^{\beta-1} \zeta(\tau) d\tau, \quad x \in [0,T], \ T > 0.$$

In particular,

$$(I^0 \zeta)(x) = \zeta(x),$$

from [1].

Let $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$. The Liouville–Caputo fractional derivative of order $\beta \in R^+$ of the function $\zeta(x)$ is defined as

$$({}_C D^\beta \zeta)(x) = I^{n-\beta} \frac{d^n}{dx^n} \zeta(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-\tau)^{n-\beta-1} \zeta^{(n)}(\tau) d\tau,$$

where $n-1 < \beta \le n$.

Assume that $\eta : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a mapping and $\phi : C[0,1] \to \mathbb{R}$ is a functional. We shall study the uniqueness, existence and stability for the following nonlinear fractional differential equation (FDE) with a nonlocal boundary condition for $1 < \beta \le 2$ and a constant λ:

$$\begin{cases} {}_C D^\beta \zeta(x) + \lambda \ {}_C D^\gamma \zeta(x) = \eta(x, \zeta(x)), \quad x \in [0,1], \\ \zeta(0) = 0, \quad \zeta(1) = \phi(\zeta), \end{cases} \tag{1.1}$$

where $0 < \gamma \le 1 < \beta$ is a constant.

Equation (1.1) with multiple terms of fractional derivatives on the left-hand side (which is different from numerous existing works) and functional boundary condition, is new and, to the best of our

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knowledge, has not been previously investigated. In order to get an implicit equivalent integral equation, we have to utilize an infinite series as an inverse operator by applying several fixed-point theorems. In addition, this equation is a generalization of various FDE with integral boundary problems.

Nonlinear boundary value problems, including nonlocal conditions, often appear in the mathematical models of real world phenomena. The study of boundary value problems is important due to their extensive applications in diverse disciplines of applied sciences and engineering. There have been many interesting investigations in the area dealing with different boundary conditions [2–17]. Adigüzel1 et al. [18] considered a boundary value problem for a nonlinear FDE with the Riemann-Liouville fractional derivative by means of a fixed point problem for an integral operator. Kumar and Malik [19] studied the existence and stability of fractional integro differential equation with noninstantaneous impulses and periodic boundary condition on time scales. Li [20] worked on the uniqueness of solutions for the following nonlinear integro-differential equation with nonlocal boundary condition and variable coefficients for $l < \alpha \leq l+1$ and $l \in \mathbb{N}$ using Banach's contractive principle:

$$\begin{cases} {}_C D^{\alpha}\zeta(x) + a(x)\, I^{\beta}\zeta(x) = g(x,\zeta(x)), \quad x \in [0,T], a \in C[0,T], \\ \zeta(0) = -\phi(\zeta), \quad \zeta''(0) = \cdots = \zeta^{(l)}(0) = 0, \\ \int_0^T \zeta(x)dx = m, \end{cases}$$

where m is a constant, $\beta \geq 0$, $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\phi : C[0,T] \to \mathbb{R}$. Very recently, Zhao et al. [21] investigated the solvability and stability of nonlinear impulsive Langevin and Sturm–Liouville equations with Caputo–Hadamard (CH) fractional derivatives and multipoint boundary value conditions. In [22], Zhao examined the solvability and generalized Ulam–Hyers (UH) stability of a nonlinear Atangana–Baleanu–Caputo (ABC) fractional coupled system with a Laplacian operator and impulses.

We define the Banach space $C[0,1]$ of all continuous functions from $[0,1]$ to $\mathbb{R}$ with the norm

$$\|\zeta\| = \max_{x\in[0,1]} |\zeta(x)| < +\infty.$$

The two-parameter Mittag–Leffler function [23] is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C}, \ \alpha, \beta > 0.$$

Babenko's approach [24] (or the inverse operator method) is a powerful tool for studying uniqueness and existence of differential equations with initial or boundary conditions. To demonstrate this in detail, we consider the following nonlinear FDE with an integral initial condition for a bounded function $g \in C([0,1] \times \mathbb{R})$:

$$\begin{cases} {}_C D^{\alpha}\zeta(x) + \lambda \ \ \zeta(x) = g(x,\zeta(x)), \quad x \in [0,1], \quad 0 < \alpha \leq 1, \\ \zeta(0) = s\displaystyle\int_0^1 \zeta(x)dx, \end{cases} \tag{1.2}$$

where s is a constant.

Applying the operator I^{α} to equation (1.2), we get

$$\zeta(x) - \zeta(0) + \lambda I^{\alpha}\zeta(x) = I^{\alpha}g(x,\zeta(x)),$$

which deduces

$$(1 + \lambda I^{\alpha})\zeta(x) = I^{\alpha}g(x,\zeta(x)) + s\int_0^1 \zeta(x)dx.$$

Then we claim that the inverse operator of $1 + \lambda I^{\alpha}$ is

$$V_{\lambda} = \sum_{k=0}^{\infty} (-1)^k \lambda^k I^{\alpha k}$$

in the space $C[0,1]$. Indeed, we have that for any $\zeta \in C[0,1]$

$$\begin{aligned} \|V_{\lambda}\zeta\| &\leq \|\zeta\| \sum_{k=0}^{\infty} |\lambda|^k \left\| I^{\alpha k} \right\| \\ &\leq \|\zeta\| \sum_{k=0}^{\infty} |\lambda|^k \frac{1}{\Gamma(\alpha k + 1)} = \|\zeta\| E_{(\alpha,1)}(|\lambda|) < +\infty, \end{aligned}$$

which implies that V_{λ} is a continuous mapping from $C[0,1]$ to itself. In addition,

$$V_{\lambda}(1 + \lambda I^{\alpha}) = (1 + \lambda I^{\alpha})V_{\lambda} = 1 \quad \text{(identity operator)}.$$

Clearly,

$$\begin{aligned} V_{\lambda}(1 + \lambda I^{\alpha}) &= \sum_{k=0}^{\infty} (-1)^k \lambda^k I^{\alpha k} + \sum_{k=0}^{\infty} (-1)^k \lambda^{k+1} I^{\alpha(k+1)} \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k \lambda^k I^{\alpha k} + \sum_{k=0}^{\infty} (-1)^k \lambda^{k+1} I^{\alpha(k+1)} \\ &= 1, \end{aligned}$$

and V_{λ} is unique.

Hence, we formally obtain

$$\begin{aligned} \zeta(x) &= (1 + \lambda I^{\alpha})^{-1} \left(I^{\alpha}g(x,\zeta(x)) + s\int_0^1 \zeta(x)dx \right) \\ &= \sum_{k=0}^{\infty} (-1)^k \lambda^k I^{\alpha k + \alpha} g(x,\zeta(x)) \\ &\quad + s\int_0^1 \zeta(x)dx \sum_{k=0}^{\infty} (-1)^k \lambda^k I^{\alpha k} 1 \\ &= \sum_{k=0}^{\infty} (-1)^k \lambda^k I^{\alpha k + \alpha} g(x,\zeta(x)) \\ &\quad + s\int_0^1 \zeta(x)dx \sum_{k=0}^{\infty} (-1)^k \lambda^k \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)}. \end{aligned}$$

The above integral equation is equivalent to Equation (1.2) with the initial condition. Furthermore, we assume that

$$d = 1 - |s| E_{(\alpha,\ 1)}(|\lambda|) > 0.$$

Then ζ is uniformly bounded on $[0,1]$. Indeed,

$$\|\zeta\| \leq \sum_{k=0}^{\infty} \frac{|\lambda|^k}{\Gamma(\alpha k + \alpha + 1)} \sup_{(x,y)\in[0,1]\times\mathbb{R}} |g(x,y)| + |s| \|\zeta\| E_{(\alpha,\ 1)}(|\lambda|).$$

Thus,

$$\|\zeta\| \le \frac{1}{d} E_{\alpha,\alpha+1}(|\lambda|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |g(x,y)| < +\infty,$$

which claims that ζ is uniformly bounded. If the function g further satisfies the following Lipschitz condition,

$$|g(x,y_1) - g(x,y_2)| \le L|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}$$

for a nonnegative constant L, and

$$Q = LE_{(\alpha,\alpha+1)}(|\lambda|) + |s|E_{(\alpha,\ 1)}(|\lambda|) < 1,$$

then Equation (1.2) has a unique solution in $C[0,1]$ by Banach's contractive principle. To show this, we start by defining a mapping $\mathbb{T}$ over $C[0,1]$ as

$$\mathbb{T}\zeta(x) = \sum_{k=0}^{\infty} (-1)^k \lambda^k I^{\alpha k+\alpha} g(x,\zeta(x)) + s\int_0^1 \zeta(x)dx \sum_{k=0}^{\infty} (-1)^k \lambda^k \frac{x^{\alpha k}}{\Gamma(\alpha k+1)}.$$

Then $\mathbb{T}\zeta \in C[0,1]$. It remains to be shown that $\mathbb{T}$ is contractive. Evidently,

$$\|\mathbb{T}\zeta_1 - \mathbb{T}\zeta_2\| \le LE_{(\alpha,\ \alpha+1)}(|\lambda|)\|\zeta_1 - \zeta_2\| + |s|E_{(\alpha,\ 1)}(|\lambda|)\|\zeta_1 - \zeta_2\| = Q\|\zeta_1 - \zeta_2\|.$$

Since $Q < 1$, Equation (1.2) has a unique solution in $C[0,1]$ from Banach's contractive principle.

In summary, we have the following theorem.

Theorem 1. *Let $g \in C([0,1]\times\mathbb{R})$ be a bounded function satisfying the Lipschitz condition:*

$$|g(x,y_1) - g(x,y_2)| \le L|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R},$$

for a nonnegative constant L. In addition, we assume that

$$Q = LE_{(\alpha,\alpha+1)}(|\lambda|) + |s|E_{(\alpha,\ 1)}(|\lambda|) < 1,$$

then Equation (1.2) has a unique solution in $C[0,1]$.

We would like to add that the method of inverse operators is also powerful in studying PDEs with initial conditions or boundary value problems. For example, Li [25] derived the following theorem making use of an inver operator:

Theorem 2. *Considering the nonlinear fractional partial integro differential equation with boundary conditions and a variable coefficient,*

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) + a(x)\, I_x^\beta u(t,x) = f(t,x,u(t,x)), & 1 < \alpha \le 2,\ \beta \ge 0, \\ u(0,x) = 0, \quad u(1,x) = \phi(x), & (t,x) \in [0,1]\times[0,1]; \end{cases}$$

we assume that a and ϕ are continuous functions over $[0,1]$, $f : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$|f(t,x,u_1) - f(t,x,u_2)| \le L|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R},$$

for a nonnegative constant L, and

$$B = L\sum_{k=0}^{\infty} \frac{\|a\|^k}{\Gamma(\alpha k+\alpha+1)\Gamma(\beta k+1)} + \frac{\|a\|}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)} + \frac{L}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)} < 1.$$

Then there exists a unique solution to the equation.

The remainder of the paper is structured as follows: In Section 2, we will derive an equivalent implicit integral equation of Equation (1.1) based on an inverse operator, then investigate the uniqueness by Banach's contractive principle with an illustrative example. Section 3 studies the existence of solutions using Leray-Schauder's fixed point theorem and presents an applicable example showing applications. Section 4 introduces a stability concept and derives sufficient conditions for equation (1.1) being stable with an example. Finally, we summarize the entire work in Section 5.

2 | Uniqueness

Theorem 3. *Let η be a continuous and bounded function on $[0,1]\times\mathbb{R}$, $\phi : C[0,1] \to \mathbb{R}$ be a functional and*

$$w = 1 - \frac{|\lambda|}{\Gamma(\beta-\gamma+1)} E_{(\beta-\gamma,\ 2)}(|\lambda|) > 0.$$

Then ζ is a solution of Equation (1.1) if and only if it satisfies the following integral equation in $C[0,1]$:

$$\begin{aligned}\zeta(x) = & \sum_{k=0}^{\infty} (-1)^k \lambda^k I^{k(\beta-\gamma)+\beta} \eta(x,\zeta(x)) \\ & + \phi(\zeta) \sum_{k=0}^{\infty} (-1)^k \lambda^k \frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ & + I_{x=1}^{\beta-\gamma} \zeta(x) \sum_{k=0}^{\infty} (-1)^k \lambda^{k+1} \frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ & - I_{x=1}^{\beta} \eta(x,\zeta(x)) \sum_{k=0}^{\infty} (-1)^k \lambda^k \frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}\end{aligned}$$

In addition,

$$\|\zeta\| \le \frac{1}{w}\left(E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \frac{1}{\gamma(\beta+1)} E_{(\beta-\gamma,\ 2)}(|\lambda|) \right) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)| + \frac{1}{w}|\phi(\zeta)| E_{(\beta-\gamma,\ 2)}(|\lambda|) < +\infty$$

Proof. Let $1 < \beta \le 2$. It follows from [23] that

$$I^\beta({}_C D^\beta)\zeta(x) = \zeta(x) - \zeta(0) - \zeta'(0)x = \zeta(x) - \zeta'(0)x,$$

using $\zeta(0) = 0$. Thus, applying the integral operator I^{β} to the equation

$$ {}_{C}D^{\beta}\zeta(x) + \lambda\ {}_{C}D^{\gamma}\zeta(x) = \eta(x, \zeta(x)), $$

we come to

$$ \zeta(x) - \zeta'(0)x + \lambda I^{\beta-\gamma}(\zeta(x) - \zeta(0)) = I^{\beta}\eta(x, \zeta(x)), $$

by noting that $0 < \gamma \le 1$. It follows from setting $x = 1$ that

$$ \phi(\zeta) - \zeta'(0) + \lambda I^{\beta-\gamma}_{x=1}\zeta(x) = I^{\beta}_{x=1}\eta(x, \zeta(x)), $$

and hence,

$$ \zeta'(0) = \phi(\zeta) + \lambda I^{\beta-\gamma}_{x=1}\zeta(x) - I^{\beta}_{x=1}\eta(x, \zeta(x)). $$

So we have

$$ \begin{aligned} &\left(1 + \lambda I^{\beta-\gamma}\right)\zeta(x) \\ &= I^{\beta}\eta(x, \zeta(x)) + x\phi(\zeta) + \lambda x I^{\beta-\gamma}_{x=1}\zeta(x) - x I^{\beta}_{x=1}\eta(x, \zeta(x)). \end{aligned} $$

Using Babenko's approach (the inverse operator), we get

$$ \begin{aligned} \zeta(x) &= \left(1 + \lambda I^{\beta-\gamma}\right)^{-1}\Big(I^{\beta}\eta(x, \zeta(x)) + x\phi(\zeta) \\ &\quad + \lambda x I^{\beta-\gamma}_{x=1}\zeta(x) - x I^{\beta}_{x=1}\eta(x, \zeta(x))\Big) \\ &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)}\Big(I^{\beta}\eta(x, \zeta(x)) + x\phi(\zeta) \\ &\quad + \lambda x I^{\beta-\gamma}_{x=1}\zeta(x) - x I^{\beta}_{x=1}\eta(x, \zeta(x))\Big) \\ &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x, \zeta(x)) \\ &\quad + \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)}x \\ &\quad + I^{\beta-\gamma}_{x=1}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1} I^{k(\beta-\gamma)}x \\ &\quad - I^{\beta}_{x=1}\eta(x, \zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)}x \\ &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x, \zeta(x)) \\ &\quad + \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k \frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &\quad + I^{\beta-\gamma}_{x=1}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1} \frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &\quad - I^{\beta}_{x=1}\eta(x, \zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k \frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}. \end{aligned} $$

Hence, ζ is a solution of Equation (1.1) if and only if it satisfies the integral Equation (2.1) since all above steps are reversible.

Furthermore,

$$ \begin{aligned} \|\zeta\| &\le \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+\beta+1)}\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x, y)| \\ &\quad + |\phi(\zeta)|\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+2)} \\ &\quad + \frac{\|\zeta\|}{\Gamma(\beta-\gamma+1)}|\lambda|\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+2)} \\ &\quad + \frac{1}{\gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+2)}\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x, y)| \\ &= E_{(\beta-\gamma,\ \beta+1)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x, y)| + |\phi(\zeta)|E_{(\beta-\gamma,\ 2)}(|\lambda|) \\ &\quad + \frac{\|\zeta\|}{\Gamma(\beta-\gamma+1)}|\lambda|E_{(\beta-\gamma,\ 2)}(|\lambda|) \\ &\quad + \frac{1}{\gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x, y)|. \end{aligned} $$

Since

$$ w = 1 - \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|) > 0, $$

we deduce

$$ \begin{aligned} \|\zeta\| &\le \frac{1}{w}\left(E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \frac{1}{\gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\right) \\ &\quad \sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x, y)| \\ &\quad + \frac{1}{w}|\phi(\zeta)|E_{(\beta-\gamma,\ 2)}(|\lambda|) < +\infty. \end{aligned} $$

This completes the proof. □

The following is the theorem regarding the uniqueness to Equation (1.1) based on Banach's contractive principle.

Theorem 4. *Let η be a continuous and bounded function on $[0, 1] \times \mathbb{R}$, satisfying the following Lipschitz condition for a nonnegative constant $\mathcal{L}_1$:*

$$ |\eta(x, y_1) - \eta(x, y_2)| \le \mathcal{L}_1|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}, $$

$\phi : C[0, 1] \to \mathbb{R}$ be a functional satisfying the condition for a nonnegative constant $\mathcal{L}_2$

$$ |\phi(\zeta_1) - \phi(\zeta_2)| \le \mathcal{L}_2\|\zeta_1 - \zeta_2\|, $$

for $\zeta_1, \zeta_2 \in C[0, 1]$. Furthermore, we assume

$$ \begin{aligned} S &= \mathcal{L}_1 E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) \\ &\quad + \left(\mathcal{L}_2 + \frac{\mathcal{L}_1}{\Gamma(\beta+1)} + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}\right)E_{(\beta-\gamma,\ 2)}(|\lambda|) < 1. \end{aligned} $$

Then Equation (1.1) has a unique solution in $C[0, 1]$.

Proof. Define a nonlinear mapping $\mathcal{M}$ over $C[0,1]$ as

$$\begin{aligned}\mathcal{M}\zeta &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x,\zeta(x)) \\ &+ \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &+ I_{x=1}^{\beta-\gamma}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &- I_{x=1}^{\beta}\eta(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.\end{aligned}$$

It follows from the proof of Theorem 3 that $\mathcal{M}\zeta \in C[0,1]$. We are going to show that $\mathcal{M}$ is contractive. Clearly,

$$\begin{aligned}\mathcal{M}\zeta_1-\mathcal{M}\zeta_2 &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}(\eta(x,\zeta_1(x))-\eta(x,\zeta_2(x))) \\ &+ (\phi(\zeta_1)-\phi(\zeta_2))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &+ I_{x=1}^{\beta-\gamma}(\zeta_1(x)-\zeta_2(x))\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &- I_{x=1}^{\beta}(\eta(x,\zeta_1(x)) \\ &- \eta(x,\zeta_2(x)))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathcal{M}\zeta_1-\mathcal{M}\zeta_2\| &\le \mathcal{L}_1\|\zeta_1-\zeta_2\|E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) \\ &+ \mathcal{L}_2\|\zeta_1-\zeta_2\|E_{(\beta-\gamma,\ 2)}(|\lambda|) \\ &+ \frac{|\lambda|}{\Gamma(\beta-\gamma+1))}\|\zeta_1-\zeta_2\|E_{(\beta-\gamma,\ 2)}(|\lambda|) \\ &+ \frac{\mathcal{L}_1}{\Gamma(\beta+1)}\|\zeta_1-\zeta_2\|E_{(\beta-\gamma,\ 2)}(|\lambda|) = S\|\zeta_1-\zeta_2\|.\end{aligned}$$

Since $S<1$, Equation (1.1) has a unique solution using Banach's contractive principle. The proof is completed. □

Example 5. The following nonlinear FDE with the nonlocal boundary condition:

$$\begin{cases} {}_CD^{1.5}\zeta(x) - \frac{1}{2}\ {}_CD^{0.5}\zeta(x) = \frac{1}{19}\sin((x^2+1)\zeta(x)) \\ \quad +\ \arctan(x^3+1), \quad x\in[0,1], \\ \zeta(0)=0, \quad \zeta(1)=\frac{1}{10(1+\zeta^2(1/2))} \end{cases} \tag{2.3}$$

has a unique solution in $C[0,1]$.

Proof. Let

$$\eta(x,\zeta) = \frac{1}{19}\sin((x^2+1)\zeta) + \arctan(x^3+1).$$

Then η is a continuous and bounded function on $[0,1]\times\mathbb{R}$, satisfying

$$\begin{aligned}|\eta(x,\zeta_1)-\eta(x,\zeta_2)| &\le \frac{1}{19}|\sin((x^2+1)\zeta_1)-\sin((x^2+1)\zeta_2)| \\ &\le \frac{2}{19}|\zeta_1-\zeta_2|,\end{aligned}$$

which infers that $\mathcal{L}_1 = 2/19$. On the other hand,

$$\phi(\zeta) = \frac{1}{10(1+\zeta^2(1/2))}$$

satisfies

$$\begin{aligned}|\phi(\zeta_1)-\phi(\zeta_2)| &\le \left|\frac{1}{10(1+\zeta_1^2(1/2))} - \frac{1}{10(1+\zeta_2^2(1/2))}\right| \\ &\le \frac{1}{10}|\zeta_1(1/2)-\zeta_2(1/2)| \le \frac{1}{10}\|\zeta_1-\zeta_2\|,\end{aligned}$$

by the mean value theorem and noting that

$$\left|\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\right| = \frac{2|x|}{(1+x^2)^2} \le 1, \quad x\in\mathbb{R}.$$

So $\mathcal{L}_2 = 1/10$ and

$$\begin{aligned}S &= \frac{2}{19}E_{(1,\ 2.5)}(1/2) \\ &+ \left(\frac{1}{10} + \frac{2/19}{\Gamma(1.5+1)} + \frac{1/2}{\Gamma(1.5-0.5+1)}\right)E_{(1,\ 2)}(1/2) \\ &= \frac{2}{19}E_{(1,\ 2.5)}(1/2) + \left(\frac{1}{10} + \frac{2}{19\,\Gamma(2.5)} + 1/2\right)E_{(1,\ 2)}(1/2) \\ &\approx \frac{2}{19} * 0.926819 + \left(\frac{1}{10} + \frac{2}{19\,\Gamma(2.5)} + 1/2\right) * 1.2974 \\ &\approx 0.978734 < 1.\end{aligned}$$

By Theorem 4, Equation (2.3) has a unique solution in $C[0,1]$. □

3 | Existence

Using Leray–Schauder's fixed point theorem, we present the following existence theorem.

Theorem 6. *Let η be a continuous and bounded function on $[0,1]\times\mathbb{R}$ and $\phi : C[0,1]\to\mathbb{R}$ be a functional satisfying the condition for a nonnegative constant $\mathcal{L}_2$*

$$|\phi(\zeta_1)-\phi(\zeta_2)| \le \mathcal{L}_2\|\zeta_1-\zeta_2\|,$$

for $\zeta_1,\zeta_2\in C[0,1]$. In addition, we assume

$$Q = 1 - \left(\mathcal{L}_2 + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}\right)E_{(\beta-\gamma,\ 2)}(|\lambda|) > 0.$$

Then there exists at least one solution to Equation (1.1) in the space $C[0,1]$.

Proof. Clearly,

$$|\phi(\zeta)| \le |\phi(\zeta)-\phi(0)| + |\phi(0)| \le \mathcal{L}_2\|\zeta\| + |\phi(0)| < +\infty,$$

if $\zeta\in C[0,1]$.

We again define the nonlinear mapping $\mathcal{M}$ over $C[0,1]$ as

$$\begin{aligned}\mathcal{M}\zeta &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x,\zeta(x))\\ &+ \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}\\ &+ I_{x=1}^{\beta-\gamma}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}\\ &- I_{x=1}^{\beta}\eta(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.\end{aligned}$$

It follows from the proof of Theorem 3 that

$$\begin{aligned}\|\mathcal{M}\zeta\| \le{}& E_{(\beta-\gamma,\ \beta+1)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x,y)| + |\phi(\zeta)|E_{(\beta-\gamma,\ 2)}(|\lambda|)\\ &+ \frac{|\lambda|\|\zeta\|}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\\ &+ \frac{1}{\gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x,y)| < +\infty\end{aligned}$$

which claims that $\mathcal{M}\zeta \in C[0,1]$. We first show that (i) $\mathcal{M}$ is continuous. In fact,

$$\begin{aligned}\|\mathcal{M}\zeta_1-\mathcal{M}\zeta_2\| \le{}& E_{(\beta-\gamma,\ \beta+1)}(|\lambda|)\sup_{x\in[0,1]}|\eta(x,\zeta_1)-\eta(x,\zeta_2)|\\ &+ \mathcal{L}_2\|\zeta_1-\zeta_2\|E_{(\beta-\gamma,\ 2)}(|\lambda|)\\ &+ \frac{|\lambda|\|\zeta_1-\zeta_2\|}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\\ &+ \frac{1}{\gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\sup_{x\in[0,1]}|\eta(x,\zeta_1)-\eta(x,\zeta_2)|.\end{aligned}$$

This implies that $\mathcal{M}$ is continuous since η is continuous.

(ii) Furthermore, we prove that $\mathcal{M}$ is a mapping from bounded sets to bounded sets. Let $\mathcal{S}$ be a bounded set in $C[0,1]$. Then for $\zeta \in \mathcal{S}$,

$$|\phi(\zeta)| = |\phi(\zeta)-\phi(0)+\phi(0)| \le \mathcal{L}_2\|\zeta\| + |\phi(0)| < C,$$

where C is a positive constant. It follows from inequality (3.1) that $\mathcal{M}\zeta$ is uniformly bounded if $\zeta \in \mathcal{S}$, as η is bounded.

(iii) We claim that $\mathcal{M}$ is completely continuous from $C[0,1]$ to itself. By the Arzela–Ascoli theorem, we need to show that $\mathcal{M}$ is equicontinuous on every bounded set $\mathcal{S}$ of $C[0,1]$. For $0 \le t_1 < t_2 \le 1$ and $\zeta \in \mathcal{S}$, we have

$$\begin{aligned}|(\mathcal{M}\zeta)(t_2)-(\mathcal{M}\zeta)(t_1)| \le{}& \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+\beta)}\\ &\cdot\left|\int_0^{t_2}(t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau\right.\\ &\left.-\int_0^{t_1}(t_1-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau\right|\\ &+ |\phi(\zeta)|\sum_{k=0}^{\infty}\frac{|\lambda^k|}{\Gamma(k(\beta-\gamma)+2)}\left|t_2^{k(\beta-\gamma)+1}-t_1^{k(\beta-\gamma)+1}\right|\\ &+ \frac{|\lambda|\|\zeta\|}{\Gamma(\beta-\gamma+1)}\sum_{k=0}^{\infty}\frac{|\lambda^k|}{\Gamma(k(\beta-\gamma)+2)}\left|t_2^{k(\beta-\gamma)+1}-t_1^{k(\beta-\gamma)+1}\right|\\ &+ \frac{1}{\Gamma(\beta+1)}\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x,y)|\\ &\sum_{k=0}^{\infty}\frac{|\lambda^k|}{\Gamma(k(\beta-\gamma)+2)}\left|t_2^{k(\beta-\gamma)+1}-t_1^{k(\beta-\gamma)+1}\right|\\ ={}& I_1+I_2+I_3+I_4.\end{aligned}$$

As for I_1, we have

$$\begin{aligned}&\int_0^{t_2}(t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau\\ &= \int_0^{t_1}(t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau\\ &\quad+ \int_{t_1}^{t_2}(t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau,\end{aligned}$$

and

$$\begin{aligned}&\int_0^{t_2}(t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau\\ &\quad-\int_0^{t_1}(t_1-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau\\ &= \int_0^{t_1}[(t_2-\tau)^{k(\beta-\gamma)+\beta-1}-(t_1-\tau)^{k(\beta-\gamma)+\beta-1}]\eta(\tau,\zeta(\tau))d\tau\\ &\quad+ \int_{t_1}^{t_2}(t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau = I_{12}+I_{22}.\end{aligned}$$

Obviously,

$$\begin{aligned}|I_{12}| &\le \int_0^{t_1}[(t_2-\tau)^{k(\beta-\gamma)+\beta-1}-(t_1-\tau)^{k(\beta-\gamma)+\beta-1}]d\tau\\ &\quad\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x,y)|\\ &= \left(-\frac{(t_2-t_1)^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}+\frac{t_2^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}-\frac{t_1^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}\right)\\ &\quad\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x,y)|\\ &\le \left(\frac{t_2^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}-\frac{t_1^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}\right)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x,y)|.\end{aligned}$$

By the mean value theorem, we deduce

$$\frac{t_2^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}-\frac{t_1^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta} = \theta^{k(\beta-\gamma)+\beta-1}(t_2-t_1) \le t_2-t_1,$$

where $t_1 < \theta < t_2$. In summary,

$$|I_{12}| \le (t_2-t_1)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\eta(x,y)|.$$

On the other hand,

$$|I_{22}| \le (t_2 - t_1) \max_{\tau \in [t_1, t_2]} |(t_2 - \tau)^{k(\beta-\gamma)+\beta-1}| \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)|$$
$$\le (t_2 - t_1) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)|.$$

Regarding I_2, I_3, and I_4, we notice that the factor

$$\left| t_2^{k(\beta-\gamma)+1} - t_1^{k(\beta-\gamma)+1} \right|$$

contains the term $t_2 - t_1$ for all $k \ge 0$. Hence, $\mathcal{M}$ is equicontinuous on every bounded set $\mathcal{S}$ of $C[0,1]$.

(iv) Finally, we will prove that the set for $0 < \delta < 1$

$$Y = \{\zeta \in C[0,1] : \zeta = \delta \mathcal{M}\zeta\}$$

is bounded. Using

$$\begin{aligned}
\|\zeta\| \le \|\mathcal{M}\zeta\| &\le E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)| + |\phi(\zeta)| E_{(\beta-\gamma,\ 2)}(|\lambda|) \\
&+ \frac{|\lambda|\|\zeta\|}{\Gamma(\beta-\gamma+1)} E_{(\beta-\gamma,\ 2)}(|\lambda|) \\
&+ \frac{1}{\gamma(\beta+1)} E_{(\beta-\gamma,\ 2)}(|\lambda|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)| \\
&\le E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)| \\
&+ (\mathcal{L}_2\|\zeta\| + |\phi(0)|) E_{(\beta-\gamma,\ 2)}(|\lambda|) \\
&+ \frac{|\lambda|\|\zeta\|}{\Gamma(\beta-\gamma+1)} E_{(\beta-\gamma,\ 2)}(|\lambda|) \\
&+ \frac{1}{\gamma(\beta+1)} E_{(\beta-\gamma,\ 2)}(|\lambda|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)|
\end{aligned}$$

and

$$\mathcal{Q} = 1 - \left(\mathcal{L}_2 + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)} \right) E_{(\beta-\gamma,\ 2)}(|\lambda|) > 0,$$

we come to

$$\begin{aligned}
\|\zeta\| \le & \frac{1}{\mathcal{Q}} \left(E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \frac{1}{\gamma(\beta+1)} E_{(\beta-\gamma,\ 2)}(|\lambda|) \right) \\
& \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\eta(x,y)| \\
& + \frac{1}{\mathcal{Q}} |\phi(0)| E_{(\beta-\gamma,\ 2)}(|\lambda|) < \infty,
\end{aligned}$$

which indicates that Y is bounded. By Leray–Schauder's fixed point theorem, Equation (1.1) has at least one solution in $C[0,1]$. This completes the proof. □

Example 7. The following nonlinear FDE with the nonlocal boundary condition:

$$\begin{cases} {}_C D^{1.8}\zeta(x) + \frac{1}{4}\ {}_C D^{0.7}\zeta(x) = \frac{x^2|\zeta(x)|}{2(1+\zeta^2(x))} + \arctan(\zeta^3(x)), & x \in [0,1], \\ \zeta(0) = 0, \quad \zeta(1) = \frac{1}{10}\sin\zeta(0.8) \end{cases} \tag{3.2}$$

has at least one solution in $C[0,1]$.

Proof. Clearly,

$$\eta(x,y) = \frac{x^2|y|}{2(1+y^2)} + \arctan y^3$$

is a continuous and bounded function on $[0,1]\times\mathbb{R}$ and

$$\phi(\zeta) = \frac{1}{10}\sin\zeta(0.8)$$

satisfies

$$\begin{aligned}
|\phi(\zeta_1) - \phi(\zeta_2)| &\le \frac{1}{10} |\sin\zeta_1(0.8) - \sin\zeta_2(0.8)| \\
&\le \frac{1}{10} |\zeta_1(0.8) - \zeta_2(0.8)| \\
&\le \frac{1}{10} \|\zeta_1 - \zeta_2\|, \quad \zeta_1,\ \zeta_2 \in C[0,1],
\end{aligned}$$

which implies that $\mathcal{L}_2 = 1/10$. We compute

$$\begin{aligned}
\mathcal{Q} &= 1 - \left(1/10 + \frac{1/4}{\Gamma(1.1+1)} \right) E_{(1.1,\ 2)}(1/4) \\
&\approx 1 - \left(1/10 + \frac{1/4}{\Gamma(1.1+1)} \right) * 1.1224 \\
&\approx 0.619677 > 0.
\end{aligned}$$

So, the equation has at least one solution in $C[0,1]$ using Theorem 6. This completes the proof. □

Remark 8. Theorem 6 does not require that the function η satisfies the Lipschitz condition. Moreover, $S < 1$ in Theorem 4 implies that $\mathcal{Q} > 0$ in Theorem 6. In addition, Equation (2.3) is handled by Theorem 4, rather than Theorem 6, since we need

$$\eta = \frac{1}{19}\sin((x^2+1)\zeta(x)) + \arctan(x^3+1),$$

to be a Lipschitz function to derive the uniqueness. However, Equation (3.2) is different as

$$\eta = \frac{x^2|\zeta(x)|}{2(1+\zeta^2(x))} + \arctan(\zeta^3(x))$$

does not meet the Lipschitz condition, but it is a continuous and bounded function on $[0,1]\times\mathbb{R}$, satisfying the condition in Theorem 6 to show the existence.

4 | Stability Analysis

Stability is an important concept in mathematics that refers to the stability of a differential equation [26]. The idea of such stability for differential equations is the substitution of the equation with a given inequality that acts as a perturbation of the equation. We are going to study the stability of Equation (1.1) based on the implicit integral Equation (2.1).

Definition 1. Equation (1.1) is stable if there exists a constant $\mathcal{K} > 0$ such that for all $\epsilon > 0$ and a continuously differentiable function ζ satisfying the boundary conditions $\zeta(0) = 0$ and $\zeta(1) = \phi(\zeta)$ and the inequality

$$\left\| {}_C D^{\beta}\zeta(x) + \lambda\ {}_C D^{\gamma}\zeta(x) - \eta(x,\zeta(x)) \right\| < \epsilon \tag{4.1}$$

then there exists a solution ζ_0 of Equation (1.1) such that

$$\|\zeta - \zeta_0\| < \mathcal{K}\epsilon,$$

where $\mathcal{K}$ is a stability constant.

Theorem 10. *Let η be a continuous and bounded function on $[0,1]\times\mathbb{R}$, satisfying the following Lipschitz condition for a nonnegative constant $\mathcal{L}_1$:*

$$|\eta(x,y_1)-\eta(x,y_2)| \le \mathcal{L}_1|y_1-y_2|, \quad y_1,y_2\in\mathbb{R},$$

$\phi : C[0,1]\to\mathbb{R}$ be a functional satisfying the condition for a nonnegative constant $\mathcal{L}_2$

$$|\phi(\zeta_1)-\phi(\zeta_2)| \le \mathcal{L}_2\|\zeta_1-\zeta_2\|,$$

for $\zeta_1,\zeta_2\in C[0,1]$. Furthermore, we assume

$$S = \mathcal{L}_1 E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \left(\mathcal{L}_2 + \frac{\mathcal{L}_1}{\Gamma(\beta+1)} + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}\right) E_{(\beta-\gamma,\ 2)}(|\lambda|) < 1.$$

Then Equation (1.1) is stable in $C[0,1]$.

Proof. We let

$$\eta_1(x,\zeta(x)) = {}_C D^{\beta}\zeta(x) + \lambda\ {}_C D^{\gamma}\zeta(x) - \eta(x,\zeta(x)),$$

which implies $\zeta(0)=0,\ \zeta(1)=\phi(\zeta)$, and

$${}_C D^{\beta}\zeta(x) + \lambda\ {}_C D^{\gamma}\zeta(x) = \eta(x,\zeta(x)) + \eta_1(x,\zeta(x)),$$

as well as

$$\|\eta_1\| < \epsilon.$$

This new equation

$$\begin{cases} {}_C D^{\beta}\zeta(x) + \lambda\ {}_C D^{\gamma}\zeta(x) = \eta(x,\zeta(x)) + \eta_1(x,\zeta(x)), & x\in[0,1], \\ \zeta(0)=0, \quad \zeta(1)=\phi(\zeta) \end{cases}$$

has at least one solution in the space $C[0,1]$ by Theorem 6, as the condition $S<1$ (in Theorem 10) implies $\mathcal{Q}>0$ in Theorem 6, and $\eta(x,\zeta(x))+\eta_1(x,\zeta(x))$ as a new function on the right-hand side of Equation (1.1) is continuous and bounded on $[0,1]\times\mathbb{R}$ (hence, all the conditions of Theorem 6 are satisfied). Furthermore, the condition $S<1$ implies $w>0$ in Theorem 3; we get by using the integral Equation (2.1) since all the conditions of Theorem 3 are also satisfied

$$\begin{aligned}\zeta(x) = &\sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}(\eta(x,\zeta(x))+\eta_1(x,\zeta(x)) \\ &+ \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &+ I^{\beta-\gamma}_{x=1}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &- I^{\beta}_{x=1}(\eta(x,\zeta(x))+\eta_1(x,\zeta(x)))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)},\end{aligned}$$

and

$$\begin{aligned}\zeta_0(x) = &\sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x,\zeta_0(x)) \\ &+ \phi(\zeta_0)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &+ I^{\beta-\gamma}_{x=1}\zeta_0(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &- I^{\beta}_{x=1}\eta(x,\zeta_0(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.\end{aligned}$$

Hence,

$$\begin{aligned}\zeta(x)-\zeta_0(x) = &\sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}(\eta(x,\zeta(x))-\eta(x,\zeta_0(x))) \\ &+ (\phi(\zeta)-\phi(\zeta_0))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &+ I^{\beta-\gamma}_{x=1}(\zeta(x)-\zeta_0(x))\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &- I^{\beta}_{x=1}(\eta(x,\zeta(x))-\eta(x,\zeta_0(x)))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} \\ &+ \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta_1(x,\zeta(x) \\ &- I^{\beta}_{x=1}\eta_1(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.\end{aligned}$$

This implies that

$$\begin{aligned}\|\zeta-\zeta_0\| \le\ &\mathcal{L}_1 E_{(\beta-\gamma,\ \beta+1)}(|\lambda|)\|\zeta-\zeta_0\| + \mathcal{L}_2 E_{(\beta-\gamma,\ 2)}(|\lambda|)\|\zeta-\zeta_0\| \\ &+ \left(\frac{\mathcal{L}_1}{\Gamma(\beta+1)} + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}\right) E_{(\beta-\gamma,\ 2)}(|\lambda|)\|\zeta-\zeta_0\| \\ &+ \left(E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\right)\epsilon,\end{aligned}$$

and

$$\|\zeta-\zeta_0\| \le \frac{1}{1-S}\left(E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\right)\epsilon = \mathcal{K}\epsilon,$$

where

$$\mathcal{K} = \frac{1}{1-S}\left(E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\right)$$

is a stability constant. Therefore, Equation (1.1) is stable in $C[0,1]$. This completes the proof. □

Remark 11. It is guaranteed that the existence of function ζ satisfying the boundary conditions $\zeta(0)=0$ and $\zeta(1)=\phi(\zeta)$ and inequality (4.1). Indeed, the following nonlinear fractional equation with any constant c

$$\begin{cases} {}_C D^{\beta}\zeta(x) + \lambda\ {}_C D^{\gamma}\zeta(x) = \eta(x,\zeta(x)) + c, & x\in[0,1], \\ \zeta(0)=0, \quad \zeta(1)=\phi(\zeta) \end{cases}$$

has at least one solution in the space $C[0,1]$ by Theorem 6, by noting that the condition $S < 1$ in Theorem 10 implies $Q > 0$ in Theorem 6, and all other conditions of Theorem 6, such as

$$|\phi(\zeta_1) - \phi(\zeta_2)| \leq \mathcal{L}_2\|\zeta_1 - \zeta_2\|, \quad \text{for} \quad \zeta_1, \zeta_2 \in C[0,1],$$

are also satisfied.

Example 12. The following nonlinear FDE with the boundary condition:

$$\begin{cases} {}_C D^{1.7}\zeta(x) - \frac{1}{4}\ {}_C D^{0.2}\zeta(x) = \frac{1}{17}\cos\left(\frac{\zeta(x)}{x^3+1}\right) + \frac{x^3 \sin x}{x^4+2}, & x \in [0,1], \\ \zeta(0) = 0, \quad \zeta(1) = 2 \end{cases} \tag{4.2}$$

is stable in $C[0,1]$.

Proof. Clearly,

$$\eta(x,\zeta) = \frac{1}{17}\cos\left(\frac{\zeta}{x^3+1}\right) + \frac{x^3 \sin x}{x^4+2}$$

is a continuous and bounded function on $[0,1] \times \mathbb{R}$, satisfying the following Lipschitz condition for $\zeta_1, \zeta_2 \in \mathbb{R}$:

$$|\eta(x,\zeta_1) - \eta(x,\zeta_2)| \leq \frac{1}{17}\left|\cos\left(\frac{\zeta_1}{x^3+1}\right) - \cos\left(\frac{\zeta_2}{x^3+1}\right)\right| \leq \frac{1}{17}|\zeta_1 - \zeta_2|,$$

which claims that $\mathcal{L}_1 = 1/17$. Furthermore, $\mathcal{L}_2 = 0$ since $\phi(\zeta) = 2$. We need to evaluate the following value by Theorem 10

$$\begin{aligned} S &= \mathcal{L}_1 E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \left(\mathcal{L}_2 + \frac{\mathcal{L}_1}{\Gamma(\beta+1)} + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}\right) E_{(\beta-\gamma,\ 2)}(|\lambda|) \\ &= \frac{1}{17}E_{(1.5,\ 2.7)}(1/4) + \left(0 + \frac{1}{17\Gamma(2.7)} + \frac{1}{4\Gamma(2.5)}\right)E_{(1.5,\ 2)}(1/4) \\ &\approx 0.283786 < 1. \end{aligned}$$

So Equation (4.2) is stable in $C[0,1]$ by Theorem 10. This completes the proof. □

In particular, for $\epsilon = 0.5$ and a continuously differentiable function ζ satisfying the boundary condition $\zeta(0) = 0,\ \zeta(1) = 2$, and the inequality

$$\left\| {}_C D^{1.7}\zeta(x) - \frac{1}{4}\ {}_C D^{0.2}\zeta(x) - \frac{1}{17}\cos\left(\frac{\zeta}{x^3+1}\right) - \frac{x^3 \sin x}{x^4+2}\right\| < 0.5,$$

then there exists a solution ζ_0 of Equation (1.1) such that

$$\|\zeta - \zeta_0\| < 0.5\mathcal{K} = 0.962205,$$

where $\mathcal{K}$ is the stability constant given by

$$\begin{aligned} \mathcal{K} &= \frac{1}{1-S}\left(E_{(\beta-\gamma,\ \beta+1)}(|\lambda|) + \frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\ 2)}(|\lambda|)\right) \\ &\approx \frac{1}{1-0.283786}\left(E_{(1.5,\ 2.7)}(1/4) + \frac{1}{\Gamma(2.7)}E_{(1.5,\ 2)}(1/4)\right) \\ &\approx 1.92441. \end{aligned}$$

5 | Conclusion

FDEs are not just mathematical curiosities—they are indispensable tools for modern science and engineering. By transcending the limitations of classical models, FDEs enable deeper insights into complex systems, drive computational innovation, and solve pressing real-world problems. In the current work, we have investigated the nonlinear FDE (1.1) with variable coefficients and functional boundary condition, which covers all types of integral boundary conditions, using the two-parameter Mittag–Leffler function, inverse operators, and fixed-point theory. In addition, we presented applicable examples making use of the theorems. Following the similar techniques, we are able to study other types of differential equations including PDEs with nonlocal boundary value problems or initial conditions. As future research, it is worth considering the following time-fractional convection–diffusion equation with an initial condition and source term for constants $a, b, \gamma \in \mathbb{R}$ by an inverse operator and the multivariate Mittage–Leffler function:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha}u(t,x) + a\ \frac{{}_c\partial^\beta}{\partial t^\beta}u(t,x) = b\ \triangle u(t,x) + \gamma\nabla u(t,x) + \phi(t,x), \\ u(0,x) = \psi(x), \end{cases}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \quad 0 < \beta < \alpha \leq 1,$

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \quad \nabla = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n},$$

and the partial Liouville–Caputo fractional derivative $\frac{{}_c\partial^\alpha}{\partial t^\alpha}$ of order $0 < \alpha \leq 1$ with respect to t is defined as

$$\left(\frac{{}_c\partial^\alpha}{\partial t^\alpha}u\right)(t,x) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}u_t'(\tau,x)d\tau.$$

Applications of the convection–diffusion equation span numerous scientific and engineering disciplines, such as fluid dynamics and heat transfer.

On the other hand, it would be interesting and challenging if we study the following equation with a variable coefficient $\lambda(x)$:

$$\begin{cases} {}_C D^\beta\zeta(x) + \lambda(x)\ {}_C D^\gamma\zeta(x) = \eta(x,\zeta(x)), & x \in [0,1], \\ \zeta(0) = 0, \quad \zeta(1) = \phi(\zeta), & 1 < \beta \leq 2, \end{cases}$$

where $0 < \gamma < \beta$ is a constant.

Author Contributions

Chenkuan Li: conceptualization, investigation, writing – original draft, methodology, validation, visualization, writing – review and editing, software, formal analysis, data curation, resources, project administration.

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Conflicts of Interest

The author declares no conflicts of interest.

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References

1. C. Li, "Several Results of Fractional Derivatives in $D'(R_+)$," *Fractional Calculus and Applied Analysis* 18 (2015): 192–207.

2. J. Tariboon, S. K. Ntouyas, and A. Singubol, "Boundary Value Problems for Fractional Differential Equations With Fractional Multiterm Integral Conditions," *Journal of Applied Mathematics* 2014 (2014): 806156.

3. R. Yan, S. Sun, Y. Sun, and Z. Han, "Boundary Value Problems for Fractional Differential Equations With Nonlocal Boundary Conditions," *Advances in Difference Equations* 2013 (2013): 176.

4. Z. Guo, M. Liu, and D. Wang, "Solutions of Nonlinear Fractional Integro-Differential Equations With Boundary Conditions," *Bull TICMI* 16 (2012): 58–65.

5. W. Sudsutad and J. Tariboon, "Existence Results of Fractional Integro-Differential Equations With M-Point Multi-Term Fractional Order Integral Boundary Conditions," *Boundary Value Problems* 2012 (2012): 94.

6. Y. Sun, Z. Zeng, and J. Song, "Existence and Uniqueness for the Boundary Value Problems of Nonlinear Fractional Differential Equations," *Applied mathematics* 8 (2017): 312–323.

7. K. Zhao, "Triple Positive Solutions for Two Classes of Delayed Nonlinear Fractional FDEs With Nonlinear Integral Boundary value Conditions," *Boundary Value Problems* 2015 (2015): 181.

8. B. Ahmad and S. Sivasundaram, "On Four-Point Nonlocal Boundary Value Problems of Nonlinear Integro-Differential Equations of Fractional Order," *Applied Mathematics and Computation* 217 (2010): 480–487.

9. S. K. Ntouyas and H. H. Al-Sulami, "A Study of Coupled Systems of Mixed Order Fractional Differential Equations and Inclusions With Coupled Integral Fractional Boundary Conditions," *Advances in Difference Equations* 2020 (2020): 73.

10. S. Meng and Y. Cui, "Multiplicity Results to a Conformable Fractional Differential Equations Involving Integral Boundary Condition," *Complexity* 2019 (2019): 8402347.

11. P. Chen and Y. Gao, "Positive Solutions for a Class of Nonlinear Fractional Differential Equations With Nonlocal Boundary Value Conditions," *Positivity* 22 (2018): 761–772.

12. A. Cabada and Z. Hamdi, "Nonlinear Fractional Differential Equations With Integral Boundary Value Conditions," *Applied Mathematics and Computation* 228 (2014): 251–257.

13. Y. Sun and M. Zhao, "Positive Solutions for a Class of Fractional Differential Equations With Integral Boundary Conditions," *Applied Mathematics Letters* 34 (2014): 17–21.

14. B. Ahmad and J. J. Nieto, "Existence Results for Nonlinear Boundary Value Problems of Fractional Integro-Differential Equations With Integral Boundary Conditions," *Boundary Value Problems* 2009 (2009): 708576.

15. C. Yang, Y. Guo, and C. Zhai, "An Integral Boundary Value Problem of Fractional Differential Equations With a Sign-Changed Parameter in Banach Spaces," *Complexity* 2021 (2021): 9567931.

16. X. H. Wang, L. P. Wang, and Q. H. Zeng, "Fractional differential equations with integral boundary conditions," *Journal of Nonlinear Sciences and Applications* 8 (2015): 309–314.

17. C. Li, R. Saadati, R. Srivastava, and J. Beaudin, "On the Boundary Value Problem of Nonlinear Fractional Integro-Differential Equations," *Mathematics* 10 (2022): 1971.

18. R. Sevinik Adigüzel, Ü. Aksoy, E. Karapinar, and İ. M. Erhan, "On the Solution of a Boundary Value Problem Associated With a Fractional Differential Equation," *Mathematical Methods in the Applied Sciences* 47 (2024): 10928–10939.

19. V. Kumar and M. Malik, "Existence and Stability of Fractional Integro Differential Equation With Non-Instantaneous Integrable Impulses and Periodic Boundary Condition on Time Scales," *Journal of King Saud University Science* 31 (2019): 1311–1317.

20. C. Li, "Uniqueness of a Nonlinear Integro-Differential Equation With Nonlocal Boundary Condition and Variable Coefficients," *Boundary Value Problems* 2023 (2023): 26.

21. K. Zhao, J. Liu, and X. Lv, "A Unified Approach to Solvability and Stability of Multipoint BVPs for Langevin and Sturm-Liouville Equations With CH Fractional Derivatives and Impulses via Coincidence Theory," *Fractal and Fractional* 8 (2024): 111.

22. K. Zhao, "Study on the Stability and Its Simulation Algorithm of a Nonlinear Impulsive ABC-Fractional Coupled System With a Laplacian Operator via F-Contractive Mapping," *Advances in Continuous and Discrete Models* 2024 (2024): 5.

23. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, 1993).

24. Y. I. Babenko, *Heat and Mass Transfer* (Khimiya, 1986) (in Russian).

25. C. Li, "On Boundary Value Problem of the Nonlinear Fractional Partial Integro-Differential Equation via Inverse Operators," *Fractional Calculus and Applied Analysis* 28 (2025): 386–410.

26. A. Mohanapriya, C. Park, A. Ganesh, and V. Govindan, "Mittag-Leffler-Hyers-Ulam Stability of Differential Equation Using Fourier Transform," *Advances in Difference Equations* 2020 (2020): 389.