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# Remarks on a New Variable-Coefficient Integro-Differential Equation via Inverse Operators

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## Abstract

In this paper, we investigate functional inverse operators associated with a class of fractional integro-differential equations. We further explore the existence, uniqueness, and stability of solutions to a new integro-differential equation featuring variable coefficients and a functional boundary condition. To demonstrate the applicability of our main theorems, we provide several examples in which we compute values of the two-parameter Mittag–Leffler functions. The proposed approach is particularly effective for addressing a wide range of integral and fractional nonlinear differential equations with initial or boundary conditions—especially those involving variable coefficients, which are typically challenging to treat using classical integral transform methods. Finally, we demonstrate a significant application of the inverse operator approach by solving a Caputo fractional convection partial differential equation in  $\mathbb{R}^n$  with an initial condition.

**Keywords:** fractional nonlinear integro-differential equation; uniqueness and existence; Banach’s contractive principle; Leray–Schauder’s fixed-point theorem; Mittag–Leffler function; functional inverse operator

**MSC:** 34B15; 34A12; 34K20; 26A33



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## 1. Introduction

The Riemann–Liouville fractional integral  $I^{\rho_1}$  of order  $\rho_1 \in \mathbb{R}^+$  is defined for the function  $H$  [1,2] as

$$(I^{\rho_1} H)(\xi) = \frac{1}{\Gamma(\rho_1)} \int_0^\xi (\xi - \sigma)^{\rho_1 - 1} H(\sigma) d\sigma.$$

It follows that

$$I^0 H = H.$$

The Liouville–Caputo fractional derivative  ${}_C D^{\rho_2}$  of order  $\rho_2 \in (1, 2]$  of the function  $H$  is defined as [1]

$$({}_C D^{\rho_2} H)(\xi) = (I^{2-\rho_2} \frac{d^2}{d\xi^2} H)(\xi) = \frac{1}{\Gamma(2-\rho_2)} \int_0^\xi (\xi - \sigma)^{1-\rho_2} H''(\sigma) d\sigma.$$

It follows that

$$I^{\rho_2} ({}_C D^{\rho_2} H)(\xi) = H(\xi) - H(0) - H'(0)\xi.$$

The set  $C[0, 1]$  is a Banach space of all continuous functions from  $[0, 1]$  into  $\mathbb{R}$  with the norm

$$\|H\| = \max_{\xi \in [0,1]} |H(\xi)| < +\infty.$$

Let  $w_1, w_2 \in C[0, 1]$  and  $w_3 : C[0, 1] \rightarrow \mathbb{R}$  be a functional. The purpose of the current work is to investigate the uniqueness, existence, and stability for the following new equation with variable coefficients for  $\gamma_j \geq 0$  ( $j = 1, 2, \dots, l \in \mathbb{N}$ ):

$$\begin{cases} {}_C D^{\rho_2} H(\xi) + w_1(\xi) I^{\rho_1} H(\xi) + \sum_{j=1}^l I^{\gamma_j} f_j(\xi, H(\xi)) = F(\xi, H(\xi)), & \xi \in [0, 1], \\ H(0) = \int_0^1 w_2(\xi) |H(\xi)| d\xi, \quad H(1) = w_3(H), \end{cases} \quad (1)$$

where  $f_j$  and  $F$  are mappings from  $[0, 1] \times \mathbb{R}$  into  $\mathbb{R}$  with certain conditions.

In addition, we aim to find a new series solution to a Caputo fractional convection equation (PDE) in  $\mathbb{R}^n$  with an initial condition based on the inverse operator method at the end to show an application of the inverse operator method in fractional PDEs.

The two-parameter Mittag-Leffler function [3] is defined by

$$E_{\beta_1, \beta_2}(\theta) = \sum_{s=0}^{\infty} \frac{\theta^s}{\Gamma(\beta_1 s + \beta_2)},$$

where  $\theta \in \mathbb{C}$ ,  $\beta_1, \beta_2 > 0$ .

To demonstrate the use of the functional inverse operator approach, we first consider the following simpler version of Equation (1) for  $0 < \rho_2 \leq 1$  and  $\rho_1 \geq 0$ :

$$\begin{cases} {}_C D^{\rho_2} H(\xi) + w_1(\xi) I^{\rho_1} H(\xi) = F(\xi, H(\xi)), & \xi \in [0, 1], \\ H(0) = \omega \int_0^1 |H(\xi)| d\xi, \end{cases} \quad (2)$$

where  $\omega$  is a constant and  $F$  is a continuous function on  $[0, 1] \times \mathbb{R}$  with

$$\|F\| = \sup_{(\xi, H) \in [0,1] \times \mathbb{R}} |F(\xi, H)| < +\infty.$$

We should point out that  $H(0) = \omega \int_0^1 |H(\xi)| d\xi$  is nonlocal because it ties the value at  $\xi = 0$  to an integral over the entire domain. Generally speaking, it appears in problems involving global feedback, delayed responses, or non-instantaneous initialization, such as heat flow with memory, ecological systems with global resource limits, and models where sensors average data over a range before feedback.

Applying  $I^{\rho_2}$  to Equation (2), we come to

$$H(\xi) + I^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) = I^{\rho_2} F(\xi, H(\xi)) + \omega \int_0^1 |H(\xi)| d\xi,$$

by noting that

$$H(0) = \omega \int_0^1 |H(\xi)| d\xi.$$

Hence,

$$(1 + I^{\rho_2} w_1(\xi) I^{\rho_1}) H(\xi) = I^{\rho_2} F(\xi, H(\xi)) + \omega \int_0^1 |H(\xi)| d\xi. \quad (3)$$

To use the inverse operator method, we first define the functional operator  $\mathcal{V}$  over the space  $C[0, 1]$  as

$$\mathcal{V} = \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s.$$

Then,  $\mathcal{V}$  is well-defined in  $C[0, 1]$ . Since, for any  $\Psi \in C[0, 1]$ , we have

$$\begin{aligned} \|\mathcal{V}\Psi\| &\leq \|\Psi\| \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2+\rho_1)}\| \leq \|\Psi\| \sum_{s=0}^{\infty} \|w_1\|^s \frac{1}{\Gamma(s(\rho_2 + \rho_1) + 1)} \\ &= \|\Psi\| E_{\rho_2+\rho_2, 1}(\|w_1\|) < +\infty. \end{aligned}$$

Thus,  $\mathcal{V}$  is continuous over  $C[0, 1]$  and the series

$$\sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s$$

is uniformly convergent. We further prove that  $\mathcal{V}$  is an inverse operator of  $1 + I^{\rho_2} w_1(\xi) I^{\rho_1}$ , namely

$$\mathcal{V}(1 + I^{\rho_2} w_1(\xi) I^{\rho_1}) = (1 + I^{\rho_2} w_1(\xi) I^{\rho_1})\mathcal{V} = 1.$$

Clearly,

$$\begin{aligned} \mathcal{V}(1 + I^{\rho_2} w_1(\xi) I^{\rho_1}) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s + \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s+1} \\ &= 1 + \sum_{s=1}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s + \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s+1} = 1. \end{aligned}$$

Moreover,  $\mathcal{V}$  is unique. In fact, assume  $\mathcal{V}_0$  is another operator satisfying

$$\mathcal{V}_0(1 + I^{\rho_2} w_1(\xi) I^{\rho_1}) = (1 + I^{\rho_2} w_1(\xi) I^{\rho_1})\mathcal{V}_0 = 1.$$

Then, we derive  $\mathcal{V}_0 = \mathcal{V}$  by applying  $\mathcal{V}$  to the above.

From Equation (3), we get

$$\begin{aligned} H(\xi) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \left( I^{\rho_2} F(\xi, H(\xi)) + \omega \int_0^1 |H(\xi)| d\xi \right) \\ &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} F(\xi, H(\xi)) \\ &\quad + \omega \int_0^1 |H(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \mathbf{1}, \end{aligned} \tag{4}$$

which is equivalent to Equation (2) and

$$H(0) = \omega \int_0^1 |H(\xi)| d\xi,$$

by noting that  $\rho_2 > 0$ .

By Banach's fixed-point theorem and the Mittag-Leffler function given above, we can use the implicit integral Equation (4) to find sufficient conditions for the uniqueness of Equation (2). To do so, we define a mapping  $\mathcal{W}$  over  $C[0, 1]$  as

$$(\mathcal{W}H)(\xi) = \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} F(\xi, H(\xi)) \\ + \omega \int_0^1 |H(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s 1.$$

Then,

$$\|\mathcal{W}H\| \leq \|F\| \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2+\rho_1)+\rho_2}\| + |\omega| \int_0^1 |H(\xi)| d\xi \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2+\rho_1)}\| \\ = \|F\| \sum_{s=0}^{\infty} \|w_1\|^s \frac{1}{\Gamma(s(\rho_2+\rho_1)+\rho_2+1)} + |\omega| \int_0^1 |H(\xi)| d\xi \sum_{s=0}^{\infty} \|w_1\|^s \frac{1}{\Gamma(s(\rho_2+\rho_1)+1)} \\ = \|F\| E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + |\omega| \int_0^1 |H(\xi)| d\xi E_{\rho_2+\rho_1, 1}(\|w_1\|) < +\infty.$$

Thus,  $\mathcal{W}$  is a mapping from  $C[0, 1]$  to itself.

Furthermore, we suppose that  $F$  satisfies the following Lipschitz condition for a constant  $\mathcal{L} \geq 0$ :

$$|F(\xi, H_1) - F(\xi, H_2)| \leq \mathcal{L}|H_1 - H_2|, \quad H_1, H_2 \in \mathbb{R},$$

and

$$\Omega = \mathcal{L} E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + |\omega| E_{\rho_2+\rho_1, 1}(\|w_1\|) < 1.$$

Then, there is a unique solution in  $C[0, 1]$  to Equation (2). We only need to show that  $\mathcal{W}$  is contractive. Clearly,

$$\mathcal{W}H_1 - \mathcal{W}H_2 \\ = \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} (F(\xi, H_1(\xi)) - F(\xi, H_2(\xi))) \\ + \omega \int_0^1 (|H_1(\xi)| - |H_2(\xi)|) d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s 1.$$

Therefore,

$$\|\mathcal{W}H_1 - \mathcal{W}H_2\| \\ \leq \mathcal{L} E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) \|H_1 - H_2\| + |\omega| E_{\rho_2+\rho_1, 1}(\|w_1\|) \|H_1 - H_2\| \\ = \Omega \|H_1 - H_2\|,$$

by noting that

$$\left| \int_0^1 (|H_1(\xi)| - |H_2(\xi)|) d\xi \right| \leq \int_0^1 |H_1(\xi) - H_2(\xi)| d\xi \leq \|H_1 - H_2\|.$$

Since  $\Omega < 1$ , by Banach's contractive principle, there exists a unique solution in  $C[0, 1]$  to Equation (2).

In summary, we have the following result:

**Theorem 1.** Let  $0 < \rho_2 \leq 1$ ,  $w_1 \in C[0,1]$ ,  $\rho_1 \geq 0$ , and  $\omega$  be a constant. We further assume that  $F(\xi, H)$  is a continuous and bounded function over  $[0,1] \times \mathbb{R}$  with the condition for a constant  $\mathcal{L} \geq 0$ :

$$|F(\xi, H_1) - F(\xi, H_2)| \leq \mathcal{L}|H_1 - H_2|, \quad H_1, H_2 \in \mathbb{R},$$

and

$$\Omega = \mathcal{L}E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + |\omega| E_{\rho_2+\rho_1, 1}(\|w_1\|) < 1.$$

Then, there is a unique solution in  $C[0,1]$  to Equation (2).

**Example 1.** The equation with a nonlocal initial condition

$$\begin{cases} {}_C D^{0.7} H(\xi) + (\xi^2 + 1) I^{0.8} H(\xi) = \frac{1}{17} \cos(H(\xi) + \xi), & \xi \in [0,1], \\ H(0) = \frac{1}{9} \int_0^1 |H(\xi)| d\xi, \end{cases} \quad (5)$$

has a unique solution in  $C[0,1]$ .

**Proof.** From the equation, we get

$$\rho_2 = 0.7, \quad \rho_1 = 0.8, \quad \|w_1\| = \max_{\xi \in [0,1]} |\xi^2 + 1| = 2, \quad \omega = 1/9,$$

and

$$F(\xi, H) = \frac{1}{17} \cos(H + \xi)$$

is continuous and bounded over  $[0,1] \times \mathbb{R}$  and satisfies

$$|F(\xi, H_1) - F(\xi, H_2)| = \frac{1}{17} |\cos(H_1 + \xi) - \cos(H_2 + \xi)| \leq \frac{1}{17} |H_1 - H_2|, \quad H_1, H_2 \in \mathbb{R},$$

which claims that  $\mathcal{L} = 1/17$ . From Theorem 1, we need to find the following value of  $\Omega$ :

$$\begin{aligned} \Omega &= \mathcal{L}E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + |\omega| E_{\rho_2+\rho_1, 1}(\|w_1\|) \\ &= \frac{1}{17} E_{1.5, 1.7}(2) + \frac{1}{9} E_{1.5, 1}(2) \approx \frac{1}{17} * 2.23841 + \frac{1}{9} * 3.3487 = 0.50374895 < 1, \end{aligned}$$

using an online calculator from Wolfram Mathematica. Therefore, there is a unique solution.  $\square$

Stability is a key topic in differential equations [4]. Stability of a differential equation generally refers to how solutions behave over time in response to (a) initial conditions, and (b) perturbations (small changes in data, inputs, or forcing terms). There are different types of stability depending on context. Our stability concept is to study solutions that do not blow up due to small errors. The idea of such stability is the substitution of a differential equation with a given inequality that acts like a perturbation of the differential equation.

**Definition 1.** Equation (1) is stable if there is a constant  $\Lambda > 0$ , such that, for all  $\epsilon > 0$  and for each fixed solution  $H \in C[0,1]$  of

$$\begin{cases} \left| {}_C D^{\rho_2} H(\xi) + w_1(\xi) I^{\rho_1} H(\xi) + \sum_{j=1}^l I^{\alpha_j} f_j(\xi, H) - F(\xi, H) \right| < \epsilon, \\ H(0) = \int_0^1 w_2(\xi) |H(\xi)| d\xi, \quad H(1) = w_3(H), \end{cases}$$

and then there exists a solution  $H_1 \in C[0, 1]$  of Equation (1), satisfying

$$\|H - H_1\| \leq \Lambda \epsilon,$$

where  $\Lambda$  is a stability constant, which is independent of  $\epsilon$ . Clearly, it is not unique.

Fractional differential equations are critical because they provide a more accurate and flexible framework for modeling real-world phenomena that exhibit memory, nonlocality, and anomalous behavior—features that classical (integer-order) differential equations cannot capture effectively. In many physical, biological, and engineering systems, the current state depends not only on the present but also on the entire history of the system. Fractional derivatives naturally incorporate memory due to their integral-based definitions (nonlocal properties). There are many interesting studies on fractional differential or integral equations due to their strong demands and applications in various pure and applied fields [5,6]. Metzler et al. [7] used fractional relaxation regarding filled polymer networks and investigated the dependence of the decisive occurring parameters on the filler content. Momani and Odibat [8] implemented analytical techniques, the variational iteration method and the Adomian decomposition method, for solving linear fractional partial differential equations arising in fluid mechanics. Dehghan and Shakeri [9] presented the solution of ordinary differential equations with multi-point boundary value conditions by means of a semi-numerical approach, which is based on the homotopy analysis method, with engineering applications. In 2021, Li [10] studied the uniqueness of solutions for certain partial integro-differential equations with the initial conditions in a Banach space using the inverse operator approach, convolution, and Banach's contractive principle. Xu et al. [11] employed the local fractional variational iteration method to obtain approximate analytical solution of the two-dimensional diffusion equation in fractal heat transfer with help of local fractional derivative and integral operators. In 2014, Cabada and Hamdi [12] studied the existence of solutions of the following nonlinear fractional differential equations with integral boundary value conditions:

$$\begin{cases} {}_{RL}D^\alpha H(\xi) + f(\xi, H(\xi)) = 0, & \xi \in (0, 1), \quad 2 < \alpha \leq 3, \\ H(0) = H'(0) = 0, \quad H(1) = \lambda \int_0^1 H(s) ds, \end{cases}$$

where  $\lambda > 0$  and  $\lambda \neq \alpha$  and  ${}_{RL}D^\alpha$  is the Riemann–Liouville fractional derivative of order  $\alpha$ .

In 2018, Zhu [13] investigated the existence and uniqueness of the following equation using the Henry–Gronwall integral inequalities:

$$\begin{cases} {}_C D^\alpha H(\xi) - {}_C D^\beta H(\xi) = F(\xi, H(\xi)), & \xi \in [0, T], \quad 0 < \beta < \alpha < 1, \quad T > 0, \\ H(0) = H_0. \end{cases}$$

Equation (1) is essentially important because it represents a new generalized fractional integro-differential boundary value problem that incorporates several key features found in complex real-world systems:

(a) It involves both a Caputo fractional derivative  ${}_C D^{\rho_2}$  and fractional integrals  $I^{\rho_1}$  as well as  $I^{\gamma_j}$ , which model memory effects and nonlocal behavior—critical in fields such as viscoelasticity, control theory, and diffusion processes.

(b) The presence of nonlinear functions  $f_j(\xi, H(\xi))$  and  $F(\xi, H(\xi))$  makes this equation more realistic for modeling physical and biological systems, where responses often depend nonlinearly on states or inputs.

(c) The variable coefficient  $w_1(\xi)$  introduces inhomogeneity, allowing the model to adapt to position-dependent properties or processes—important in heterogeneous media or materials with spatial variation.

(d) The boundary conditions are non-standard: At  $\xi = 0$ , the value depends on an integral condition involving  $w_2(\xi)$ . At  $\xi = 1$ , it depends on a functional  $w_3(H)$ , which could be nonlocal, nonlinear, or global in nature. These types of boundary conditions arise naturally in population dynamics, thermodynamics, economics, and fluid mechanics, where the state at a boundary is governed by average or cumulative behavior.

(e) From a theoretical perspective, studying existence, uniqueness, and stability for such an equation is nontrivial due to the combination of fractional operators, nonlinear terms, functional boundary conditions, and a mixed initial-boundary structure.

The motivation for employing the inverse operator method and Mittag–Leffler functions in the present work stems from the fact that, to the best of our knowledge, there are no existing integral transforms or alternative approaches capable of converting Equation (1) into an equivalent integral equation. However, to apply fixed-point theory for studying uniqueness and existence, such an equivalent integral formulation is essential in order to define a suitable nonlinear operator.

In the following sections, we derive an implicit integral equation that is equivalent to Equation (1). Then, by Banach’s contractive principle, a functional inverse operator, as well as the Mittag–Leffler function, we will obtain sufficient conditions for the uniqueness and stability. Furthermore, we study the existence based on Leray–Schauder’s fixed-point theorem. Also included are illustrative examples demonstrating applications of the main theorems. At the end, we find a series solution to a time-fractional convection equation in  $\mathbb{R}^n$  to show an application of the inverse operator method in PDEs.

## 2. Uniqueness and Stability

**Theorem 2.** Let  $w_1, w_2 \in C[0, 1]$ ,  $w_3 : C[0, 1] \rightarrow \mathbb{R}$  be a functional,  $1 < \rho_2 \leq 2$  and  $\rho_1, \gamma_j \geq 0$ . Furthermore, we assume that  $F, f_j$  are continuous and bounded functions over  $[0, 1] \times \mathbb{R}$ . Then, Equation (1) is equivalent to the implicit integral equation in  $C[0, 1]$ :

$$\begin{aligned}
 H(\xi) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} F(\xi, H(\xi)) \\
 &\quad - \sum_{j=1}^l \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \\
 &\quad + w_3(H) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi + I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad + \sum_{j=1}^l I_{\xi=1}^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad - I_{\xi=1}^{\rho_2} F(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad + \int_0^1 w_2(\xi) |H(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s (1 - \xi).
 \end{aligned} \tag{6}$$

In addition, if

$$Q = 1 - \left( \frac{\|w_1\|}{\Gamma(\rho_2 + 1)\Gamma(\rho_1 + 1)} + \|w_2\| \right) E_{\rho_2 + \rho_1, 1}(\|w_1\|) > 0,$$

then

$$\begin{aligned} \|H\| \leq & \frac{\|F\|}{Q} E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + \frac{1}{Q} \sum_{j=1}^l \|f_j\| E_{\rho_2+\rho_1, \gamma_j+\rho_2+1}(\|w_1\|) \\ & + \frac{|w_3(H)|}{Q} E_{\rho_2+\rho_1, 1}(\|w_1\|) \\ & + \frac{1}{Q} E_{\rho_2+\rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{\|f_j\|}{\Gamma(\gamma_j + \rho_2 + 1)} + \frac{\|F\|}{Q \Gamma(\rho_2 + 1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) < +\infty. \end{aligned}$$

**Proof.** Part I: Apply  $I^{\rho_2}$  to Equation (1) to get

$$H(\xi) + c_1 + c_2\xi + I^{\rho_2}w_1(\xi)I^{\rho_1}H(\xi) + \sum_{j=1}^l I^{\alpha_j+\rho_2}f_j(\xi, H(\xi)) = I^{\rho_2}F(\xi, H(\xi)). \quad (7)$$

This claims

$$c_1 = -H(0) = -\int_0^1 w_2(\xi)|H(\xi)|d\xi,$$

by setting  $\xi = 0$ . Evidently,

$$I_{\xi=1}^{\rho_2}w_1(\xi)I^{\rho_1}H(\xi) = \frac{1}{\Gamma(\rho_2)\Gamma(\rho_1)} \int_0^1 (1-\sigma)^{\rho_2-1}w_1(\sigma)d\sigma \int_0^\sigma (\sigma-\tau)^{\rho_1-1}H(\tau)d\tau$$

is a constant for  $H \in C[0, 1]$ , and, from  $\xi = 1$ , we get

$$\begin{aligned} c_2 = & -w_3(H) - I_{\xi=1}^{\rho_2}w_1(\xi)I^{\rho_1}H(\xi) - \sum_{j=1}^l I_{\xi=1}^{\gamma_j+\rho_2}f_j(\xi, H(\xi)) \\ & + I_{\xi=1}^{\rho_2}F(\xi, H(\xi)) + \int_0^1 w_2(\xi)|H(\xi)|d\xi. \end{aligned}$$

Hence,

$$\begin{aligned} & (1 + I^{\rho_2}w_1(\xi)I^{\rho_1})H(\xi) \\ & = I^{\rho_2}F(\xi, H(\xi)) - \sum_{j=1}^l I^{\gamma_j+\rho_2}f_j(\xi, H(\xi)) + \xi w_3(H) + \xi I_{\xi=1}^{\rho_2}w_1(\xi)I^{\rho_1}H(\xi) \\ & + \xi \sum_{j=1}^l I_{\xi=1}^{\gamma_j+\rho_2}f_j(\xi, H(\xi)) - \xi I_{\xi=1}^{\rho_2}F(\xi, H(\xi)) + (1-\xi) \int_0^1 w_2(\xi)|H(\xi)|d\xi. \end{aligned}$$

From Section 1, we can similarly show that the inverse operator of  $1 + I^{\rho_2}w_1(x)I^{\rho_1}$  is

$$V = \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2}w_1(\xi)I^{\rho_1})^k$$

in the space  $C[0, 1]$ . Indeed,

$$\|V\| \leq \sum_{s=0}^{\infty} \|w_1\|^k \|I^{(\rho_1+\rho_2)k}\| \leq \sum_{s=0}^{\infty} \|w_1\|^k \frac{1}{\Gamma((\rho_1+\rho_2)k+1)} = E_{\rho_1+\rho_2, 1}(\|w_1\|) < +\infty.$$

Moreover,

$$V(1 + I^{\rho_2}w_1(x)I^{\rho_1}) = (1 + I^{\rho_2}w_1(x)I^{\rho_1})V = 1,$$

and  $V$  is unique.

So,

$$\begin{aligned}
 H(\xi) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} F(\xi, H(\xi)) \\
 &\quad - \sum_{j=1}^l \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \\
 &\quad + w_3(H) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi + I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad + \sum_{j=1}^l I_{\xi=1}^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad - I_{\xi=1}^{\rho_2} F(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad + \int_0^1 w_2(\xi) |H(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s (1 - \xi).
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 \|H\| &\leq \|F\| \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2 + \rho_1) + \rho_2}\| + \sum_{j=1}^l \|f_j\| \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2 + \rho_1) + \gamma_j + \rho_2}\| \\
 &\quad + |w_3(H)| \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2 + \rho_1)}\| + \frac{\|w_1\| \|H\|}{\Gamma(\rho_2 + 1) \Gamma(\rho_1 + 1)} \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2 + \rho_1)}\| \\
 &\quad + \sum_{j=1}^l \frac{\|f_j\|}{\Gamma(\gamma_j + \rho_2 + 1)} \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2 + \rho_1)}\| + \frac{\|F\|}{\Gamma(\rho_2 + 1)} \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2 + \rho_1)}\| \\
 &\quad + \|w_2\| \|H\| \sum_{s=0}^{\infty} \|w_1\|^s \|I^{s(\rho_2 + \rho_1)}\| \\
 &\leq \|F\| E_{\rho_2 + \rho_1, \rho_2 + 1}(\|w_1\|) + \sum_{j=1}^l \|f_j\| E_{\rho_2 + \rho_1, \gamma_j + \rho_2 + 1}(\|w_1\|) + |w_3(H)| E_{\rho_2 + \rho_1, 1}(\|w_1\|) \\
 &\quad + \frac{\|w_1\| \|H\|}{\Gamma(\rho_2 + 1) \Gamma(\rho_1 + 1)} E_{\rho_2 + \rho_1, 1}(\|w_1\|) + E_{\rho_2 + \rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{\|f_j\|}{\Gamma(\gamma_j + \rho_2 + 1)} \\
 &\quad + \frac{\|F\|}{\Gamma(\rho_2 + 1)} E_{\rho_2 + \rho_1, 1}(\|w_1\|) + \|w_2\| \|H\| E_{\rho_2 + \rho_1, 1}(\|w_1\|).
 \end{aligned}$$

Since

$$Q = 1 - \left( \frac{\|w_1\|}{\Gamma(\rho_2 + 1) \Gamma(\rho_1 + 1)} + \|w_2\| \right) E_{\rho_2 + \rho_1, 1}(\|w_1\|) > 0,$$

we get

$$\begin{aligned}
 \|H\| &\leq \frac{\|F\|}{Q} E_{\rho_2 + \rho_1, \rho_2 + 1}(\|w_1\|) + \frac{1}{Q} \sum_{j=1}^l \|f_j\| E_{\rho_2 + \rho_1, \gamma_j + \rho_2 + 1}(\|w_1\|) \\
 &\quad + \frac{|w_3(H)|}{Q} E_{\rho_2 + \rho_1, 1}(\|w_1\|) \\
 &\quad + \frac{1}{Q} E_{\rho_2 + \rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{\|f_j\|}{\Gamma(\gamma_j + \rho_2 + 1)} + \frac{\|F\|}{Q \Gamma(\rho_2 + 1)} E_{\rho_2 + \rho_1, 1}(\|w_1\|) < +\infty.
 \end{aligned}$$

Part II: To prove that  $H(\xi)$  given in Formula (6) is a solution of Equation (1), we see that  $V$  is a unique inverse operator of  $1 + I^{\rho_2} w_1(x) I^{\rho_1}$ . This implies  $H(\xi)$  satisfies Equation (7), which is equivalent to Equation (6):

$$H(\xi) + c_1 + c_2 \xi + I^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) + I^{\rho_2} \sum_{j=1}^l I^{\alpha_j} f_j(\xi, H(\xi)) = I^{\rho_2} F(\xi, H(\xi)).$$

Using Lemma 2.21 (a) in [2], we apply the operator  ${}_C D^{\rho_2}$  to both sides of the above equation to get

$${}_C D^{\rho_2} H(\xi) + w_1(\xi) I^{\rho_1} H(\xi) + \sum_{j=1}^l I^{\gamma_j} f_j(\xi, H(\xi)) = F(\xi, H(\xi)).$$

Clearly, it satisfies the boundary conditions by reversing the above steps in Part I. This completes the proof.  $\square$

Regarding the uniqueness and stability, we have the theorem below.

**Theorem 3.** Let  $w_1, w_2 \in C[0, 1]$ ,  $1 < \rho_2 \leq 2$ , and  $\rho_1, \gamma_j \geq 0$ . Furthermore, we assume that  $F, f_j$  are continuous and bounded functions over  $[0, 1] \times \mathbb{R}$ , with conditions

$$\begin{aligned} |F(\xi, H_1) - F(\xi, H_2)| &\leq F_0 |H_1 - H_2|, \quad H_1, H_2 \in \mathbb{R}, \quad \text{and} \\ |f_j(\xi, H_1) - f_j(\xi, H_2)| &\leq F_j |H_1 - H_2|, \quad H_1, H_2 \in \mathbb{R}. \end{aligned}$$

Let  $w_3 : C[0, 1] \rightarrow \mathbb{R}$  be a functional such that

$$|w_3(H_1) - w_3(H_2)| \leq \mathcal{C} \|H_1 - H_2\|.$$

If

$$\begin{aligned} A = & F_0 E_{\rho_2 + \rho_1, \rho_2 + 1}(\|w_1\|) + \sum_{j=1}^l F_j E_{\rho_2 + \rho_1, \gamma_j + \rho_2 + 1}(\|w_1\|) + \mathcal{C} E_{\rho_2 + \rho_1, 1}(\|w_1\|) \\ & + \frac{\|w_1\|}{\Gamma(\rho_2 + 1) \Gamma(\rho_1 + 1)} E_{\rho_2 + \rho_1, 1}(\|w_1\|) + E_{\rho_2 + \rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{F_j}{\Gamma(\gamma_j + \rho_2 + 1)} \\ & + \frac{F_0}{\Gamma(\rho_2 + 1)} E_{\rho_2 + \rho_1, 1}(\|w_1\|) + \|w_2\| E_{\rho_2 + \rho_1, 1}(\|w_1\|) < 1, \end{aligned}$$

then Equation (1) is stable with a unique solution in  $C[0, 1]$ .

**Proof.** To show the uniqueness, we use a mapping  $\mathbb{M}$  over the space  $C[0, 1]$ , given by

$$\begin{aligned} (\mathbb{M}H)(\xi) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} F(\xi, H(\xi)) \\ &\quad - \sum_{j=1}^l \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \\ &\quad + w_3(H) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi + I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &\quad + \sum_{j=1}^l I_{\xi=1}^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &\quad - I_{\xi=1}^{\rho_2} F(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &\quad + \int_0^1 w_2(\xi) |H(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s (1 - \xi). \end{aligned}$$

From Theorem 2, the mapping  $\mathbb{M}$  is from  $C[0, 1]$  to itself. We will prove that  $\mathbb{M}$  is contractive. For  $H_1, H_2 \in C[0, 1]$ ,

$$\begin{aligned} &\mathbb{M}H_1 - \mathbb{M}H_2 \\ &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} (F(\xi, H_1(\xi)) - F(\xi, H_2(\xi))) \\ &\quad - \sum_{j=1}^l \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\gamma_j + \rho_2} (f_j(\xi, H_1(\xi)) - f_j(\xi, H_2(\xi))) \\ &\quad + (w_3(H_1) - w_3(H_2)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &\quad + I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} (H_1(\xi) - H_2(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &\quad + \sum_{j=1}^l I_{\xi=1}^{\gamma_j + \rho_2} (f_j(\xi, H_1(\xi)) - f_j(\xi, H_2(\xi))) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &\quad - I_{\xi=1}^{\rho_2} (F(\xi, H_1(\xi)) - F(\xi, H_2(\xi))) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &\quad + \int_0^1 w_2(\xi) (|H_1(\xi)| - |H_2(\xi)|) d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s (1 - \xi). \end{aligned}$$

This implies

$$\begin{aligned} \|\mathbb{M}H_1 - \mathbb{M}H_2\| &\leq F_0 E_{\rho_2 + \rho_1, \rho_2 + 1} (\|w_1\|) \|H_1 - H_2\| \\ &\quad + \sum_{j=1}^l F_j E_{\rho_2 + \rho_1, \gamma_j + \rho_2 + 1} (\|w_1\|) \|H_1 - H_2\| \\ &\quad + \mathcal{C} E_{\rho_2 + \rho_1, 1} (\|w_1\|) \|H_1 - H_2\| + \frac{\|w_1\|}{\Gamma(\rho_2 + 1) \Gamma(\rho_1 + 1)} E_{\rho_2 + \rho_1, 1} (\|w_1\|) \|H_1 - H_2\| \\ &\quad + E_{\rho_2 + \rho_1, 1} (\|w_1\|) \sum_{j=1}^l \frac{F_j}{\Gamma(\gamma_j + \rho_2 + 1)} \|H_1 - H_2\| + \frac{F_0}{\Gamma(\rho_2 + 1)} E_{\rho_2 + \rho_1, 1} (\|w_1\|) \|H_1 - H_2\| \\ &\quad + \|w_2\| E_{\rho_2 + \rho_1, 1} (\|w_1\|) \|H_1 - H_2\| = A \|H_1 - H_2\|. \end{aligned}$$

Since  $A < 1$ , by Banach’s contractive principle, there is a unique solution in  $C[0, 1]$ .

To prove the stability, we let

$$\zeta_H(\xi) = {}_C D^{\rho_2} H(\xi) + w_1(\xi) I^{\rho_1} H(\xi) + \sum_{j=1}^l I^{\gamma_j} f_j(\xi, H(\xi)) - F(\xi, H(\xi)).$$

Then,  $\|\zeta_H\| < \epsilon$  and

$$\begin{cases} {}_C D^{\rho_2} H(\xi) + w_1(\xi) I^{\rho_1} H(\xi) + \sum_{j=1}^l I^{\gamma_j} f_j(\xi, H(\xi)) = F(\xi, H(\xi)) + \zeta_H(x), & x \in [0, 1], \\ H(0) = \int_0^1 w_2(\xi) |H(\xi)| d\xi, & H(1) = w_3(H). \end{cases} \tag{8}$$

Since  $f(\xi, H(\xi)) + \zeta_H(\xi)$  is continuous and bounded, we have from Theorem 2

$$\begin{aligned} H(\xi) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} (F(\xi, H(\xi)) + \zeta_H(\xi)) \\ &- \sum_{j=1}^l \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \\ &+ w_3(H) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi + I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &+ \sum_{j=1}^l I_{\xi=1}^{\gamma_j + \rho_2} f_j(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &- I_{\xi=1}^{\rho_2} (F(\xi, H(\xi)) + \zeta_H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &+ \int_0^1 w_2(\xi) |H(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s (1 - \xi). \end{aligned}$$

On the other hand, we get from the uniqueness of Equation (1) that

$$\begin{aligned} H_1(\xi) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} F(\xi, H_1(\xi)) \\ &- \sum_{j=1}^l \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\gamma_j + \rho_2} f_j(\xi, H_1(\xi)) \\ &+ w_3(H_1) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi + I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} H_1(\xi) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &+ \sum_{j=1}^l I_{\xi=1}^{\gamma_j + \rho_2} f_j(\xi, H_1(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &- I_{\xi=1}^{\rho_2} F(\xi, H_1(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\ &+ \int_0^1 w_2(\xi) |H_1(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s (1 - \xi). \end{aligned}$$

Then,

$$\|H - H_1\| \leq A \|H - H_1\| + E_{\rho_2 + \rho_1, \rho_2 + 1}(\|w_1\|) \|\zeta_H\| + \frac{\|\zeta_H\|}{\Gamma(\rho_2 + 1)} E_{\rho_2 + \rho_1, 1}(\|w_1\|),$$

which deduces

$$\|H - H_1\| \leq \frac{E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|)}{1-A} \|\zeta_H\| + \frac{\|\zeta_H\|}{(1-A)\Gamma(\rho_2+1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) \leq \Lambda \epsilon,$$

where

$$\Lambda = \frac{E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|)}{1-A} + \frac{E_{\rho_2+\rho_1, 1}(\|w_1\|)}{\Gamma(\rho_2+1)(1-A)}$$

is a stability constant, which is independent of  $\epsilon$ .  $\square$

**Example 2.** *The equation with a variable coefficient*

$$\begin{cases} {}_C D^{1.5} H(\xi) + \frac{1}{19(\xi+1)} I^{1.1} H(\xi) + \frac{1}{31} I^{0.8} \cos(H(\xi)\xi^2) \\ \quad = \frac{1}{21} \sin(H(\xi) + \xi^3), \quad \xi \in [0, 1], \\ H(0) = \frac{1}{13} \int_0^1 \xi |H(\xi)| d\xi, \quad H(1) = \frac{1}{15} \arctan H(1/2), \end{cases} \quad (9)$$

is stable and has a unique solution in  $C[0, 1]$ .

**Proof.** From the equation, we can see that

$$\rho_2 = 1.5, \quad \|w_1\| = 1/19, \quad \rho_1 = 1.1, \quad \|w_2\| = 1/13, \quad \gamma_1 = 0.8,$$

and

$$F(\xi, H) = \frac{1}{21} \sin(H + \xi^3)$$

is a continuous and bounded function, and

$$|F(\xi, H_1) - F(\xi, H_2)| \leq \frac{1}{21} |\sin(H_1 + \xi^3) - \sin(H_2 + \xi^3)| \leq \frac{1}{21} |H_1 - H_2|,$$

which indicates  $F = 1/21$ . Similarly,

$$f_1(\xi, H) = \frac{1}{31} \cos(H\xi^2)$$

is a continuous and bounded function with

$$|f_1(\xi, H_1) - f_1(\xi, H_2)| \leq \frac{1}{31} |H_1 - H_2|,$$

which claims  $F_1 = 1/31$ , and

$$w_3(H) = \frac{1}{15} \arctan H(1/2)$$

satisfies

$$|w_3(H_1) - w_3(H_2)| \leq \frac{1}{15} |H_1(1/2) - H_2(1/2)| \leq \frac{1}{15} \|H_1 - H_2\|,$$

which holds that  $C = 1/15$ . We need to evaluate the value of

$$\begin{aligned}
 A &= F_0 E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + \sum_{j=1}^l F_j E_{\rho_2+\rho_1, \gamma_j+\rho_2+1}(\|w_1\|) + C E_{\rho_2+\rho_1, 1}(\|w_1\|) \\
 &\quad + \frac{\|w_1\|}{\Gamma(\rho_2+1)\Gamma(\rho_1+1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) + E_{\rho_2+\rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{F_j}{\Gamma(\gamma_j+\rho_2+1)} \\
 &\quad + \frac{F_0}{\Gamma(\rho_2+1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) + \|w_2\| E_{\rho_2+\rho_1, 1}(\|w_1\|) \\
 &= \frac{1}{21} E_{2.6, 2.5}(1/19) + \frac{1}{31} E_{2.6, 3.3}(1/19) + \left( \frac{1}{15} + \frac{1}{19\Gamma(2.5)\Gamma(2.1)} \right. \\
 &\quad \left. + \frac{1}{31\Gamma(3.3)} + \frac{1}{21\Gamma(2.5)} + \frac{1}{13} \right) E_{2.6, 1}(1/19) \\
 &\approx \frac{1}{21} * 0.754138 + \frac{1}{31} * 0.373176 + 0.229266 * 1.01418 \\
 &= 0.280466260697 < 1.
 \end{aligned}$$

By Theorem 3, Equation (9) is stable with a unique solution.  $\square$

### 3. Existence

We will prove the theorem about the existence of Equation (1) using Leray–Schauder’s fixed-point theorem.

**Theorem 4.** Let  $w_1, w_2 \in C[0, 1]$ ,  $1 < \rho_2 \leq 2$ , and  $\rho_1, \gamma_j \geq 0$  for all  $j = 1, \dots, l$ . Furthermore, we assume that  $F, f_j$  are continuous and bounded functions over  $[0, 1] \times \mathbb{R}$ , and  $w_3 : C[0, 1] \rightarrow \mathbb{R}$  is a functional with the condition

$$|w_3(H_1) - w_3(H_2)| \leq C \|H_1 - H_2\|,$$

for a nonnegative constant  $C$ , and

$$Q_0 = 1 - \left( \frac{\|w_1\|}{\Gamma(\rho_2+1)\Gamma(\rho_1+1)} + \|w_2\| + C \right) E_{\rho_2+\rho_1, 1}(\|w_1\|) > 0.$$

Then, there is at least one solution to Equation (1).

**Proof.** We use the mapping  $\mathbb{M}$  again, given by

$$\begin{aligned}
 (\mathbb{M}H)(\xi) &= \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\rho_2} F(\xi, H(\xi)) \\
 &\quad - \sum_{j=1}^l \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s I^{\gamma_j+\rho_2} f_j(\xi, H(\xi)) \\
 &\quad + w_3(H) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi + I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad + \sum_{j=1}^l I_{\xi=1}^{\gamma_j+\rho_2} f_j(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad - I_{\xi=1}^{\rho_2} F(\xi, H(\xi)) \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s \xi \\
 &\quad + \int_0^1 w_2(\xi) |H(\xi)| d\xi \sum_{s=0}^{\infty} (-1)^s (I^{\rho_2} w_1(\xi) I^{\rho_1})^s (1 - \xi).
 \end{aligned}$$

From Theorem 2,  $\mathbb{M}$  is a mapping from  $C[0, 1]$  to itself. For any  $H_1, H_2 \in C[0, 1]$ , we have

$$\begin{aligned} \|\mathbb{M}H_1 - \mathbb{M}H_2\| &\leq \sup_{\xi \in [0,1]} |F(\xi, H_1(\xi)) - F(\xi, H_2(\xi))| E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) \\ &+ \sum_{j=1}^l \sup_{\xi \in [0,1]} |f_j(\xi, H_1(\xi)) - f_j(\xi, H_2(\xi))| E_{\rho_2+\rho_1, \gamma_j+\rho_2+1}(\|w_1\|) \\ &+ \mathcal{C} E_{\rho_2+\rho_1, 1}(\|w_1\|) \|H_1 - H_2\| + \frac{\|w_1\|}{\Gamma(\rho_2+1)\Gamma(\rho_1+1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) \|H_1 - H_2\| \\ &+ E_{\rho_2+\rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{\sup_{\xi \in [0,1]} |f_j(\xi, H_1(\xi)) - f_j(\xi, H_2(\xi))|}{\Gamma(\gamma_j + \rho_2 + 1)} \\ &+ \frac{\sup_{\xi \in [0,1]} |F(\xi, H_1(\xi)) - F(\xi, H_2(\xi))|}{\Gamma(\rho_2 + 1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) \\ &+ \|w_2\| E_{\rho_2+\rho_1, 1}(\|w_1\|) \|H_1 - H_2\|, \end{aligned}$$

which implies that (i)  $\mathbb{M}$  is continuous because  $F$  and  $f_j$  are continuous.

(ii)  $\mathbb{M}$  is a mapping from bounded sets in  $C[0, 1]$  to bounded sets. Assume  $\mathcal{B}$  is a bounded set in  $C[0, 1]$ . For any  $H \in \mathcal{B}$ ,

$$|w_3(H)| = |w_3(H) - w_3(0) + w_3(0)| \leq \mathcal{C} \|H\| + |w_3(0)|$$

is bounded. Using

$$\begin{aligned} \|\mathbb{M}H\| &\leq \|F\| E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + \sum_{j=1}^l \|f_j\| E_{\rho_2+\rho_1, \gamma_j+\rho_2+1}(\|w_1\|) \\ &+ |w_3(H)| E_{\rho_2+\rho_1, 1}(\|w_1\|) \\ &+ \frac{\|w_1\| \|H\|}{\Gamma(\rho_2+1)\Gamma(\rho_1+1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) + E_{\rho_2+\rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{\|f_j\|}{\Gamma(\gamma_j + \rho_2 + 1)} \\ &+ \frac{\|F\|}{\Gamma(\rho_2+1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) + \|w_2\| \|H\| E_{\rho_2+\rho_1, 1}(\|w_1\|), \end{aligned}$$

one claims the set

$$\{\mathbb{M}\chi : \chi \in \mathcal{B}\}$$

is bounded by noting that  $F$  and  $f_j$  are bounded.

(iii)  $\mathbb{M}H$  is equicontinuous over  $[0, 1]$  for  $H \in \mathcal{B}$ . Let  $0 \leq \xi_1 < \xi_2 \leq 1$ , and we have

$$\begin{aligned}
 \mathbb{M}H(\xi_2) - \mathbb{M}H(\xi_1) &= I_{\xi=\xi_2}^{\rho_2} F(\xi, H(\xi)) - I_{\xi=\xi_1}^{\rho_2} F(\xi, H(\xi)) \\
 &+ \sum_{s=1}^{\infty} (-1)^s \left[ I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} I^{\rho_2} F(\xi, H(\xi)) \right. \\
 &- \left. I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} I^{\rho_2} F(\xi, H(\xi)) \right] \quad (= T_1) \\
 &- \sum_{j=1}^l \left[ I_{\xi=\xi_2}^{\gamma_j+\rho_2} f_j(\xi, H(\xi)) - I_{\xi=\xi_1}^{\gamma_j+\rho_2} f_j(\xi, H(\xi)) \right] \\
 &- \sum_{j=1}^l \sum_{s=1}^{\infty} (-1)^s \left[ I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} I^{\gamma_j+\rho_2} f_j(\xi, H(\xi)) \right. \\
 &- \left. I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} I^{\gamma_j+\rho_2} f_j(\xi, H(\xi)) \right] \quad (= T_2) \\
 &+ w_3(H) \left[ (\xi_2 - \xi_1) + \sum_{s=1}^{\infty} (-1)^s \left( I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right. \right. \\
 &- \left. \left. I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right) \right] \quad (= T_3) \\
 &+ I_{\xi=1}^{\rho_2} w_1(\xi) I^{\rho_1} H(\xi) \left[ \xi_2 - \xi_1 + \sum_{s=1}^{\infty} (-1)^s \left( I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right. \right. \\
 &- \left. \left. I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right) \right] \quad (= T_4) \\
 &+ \sum_{j=1}^l I_{\xi=1}^{\gamma_j+\rho_2} f_j(\xi, H(\xi)) \left[ \xi_2 - \xi_1 + \sum_{s=1}^{\infty} (-1)^s \left( I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right. \right. \\
 &- \left. \left. I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right) \right] \quad (= T_5) \\
 &- I_{\xi=1}^{\rho_2} F(\xi, H(\xi)) \left[ \xi_2 - \xi_1 + \sum_{s=1}^{\infty} (-1)^s \left( I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right. \right. \\
 &- \left. \left. I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} \xi \right) \right] \quad (= T_6) \\
 &+ \int_0^1 w_2(\xi) |H(\xi)| d\xi \left[ (1 - \xi_2) - (1 - \xi_1) + \sum_{s=1}^{\infty} (-1)^s \left( I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} (1 - \xi) \right. \right. \\
 &- \left. \left. I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} (1 - \xi) \right) \right] \quad (= T_7).
 \end{aligned}$$

We are going to prove  $T_1$  is a Lipschitz function. Indeed,

$$\begin{aligned}
 &I_{\xi=\xi_2}^{\rho_2} F(\xi, H(\xi)) - I_{\xi=\xi_1}^{\rho_2} F(\xi, H(\xi)) \\
 &= \frac{1}{\Gamma(\rho_2)} \int_0^{\xi_1} (\xi_2 - \sigma)^{\rho_2-1} F(\sigma, H(\sigma)) d\sigma + \frac{1}{\Gamma(\rho_2)} \int_{\xi_1}^{\xi_2} (\xi_2 - \sigma)^{\rho_2-1} F(\sigma, H(\sigma)) d\sigma \\
 &- \frac{1}{\Gamma(\rho_2)} \int_0^{\xi_1} (\xi_1 - \sigma)^{\rho_2-1} F(\sigma, H(\sigma)) d\sigma \\
 &= \frac{1}{\Gamma(\rho_2)} \int_0^{\xi_1} \left( (\xi_2 - \sigma)^{\rho_2-1} - (\xi_1 - \sigma)^{\rho_2-1} \right) F(\sigma, H(\sigma)) d\sigma \\
 &+ \frac{1}{\Gamma(\rho_2)} \int_{\xi_1}^{\xi_2} (\xi_2 - \sigma)^{\rho_2-1} F(\sigma, H(\sigma)) d\sigma.
 \end{aligned}$$

Clearly,

$$\frac{1}{\Gamma(\rho_2)} \left| \int_{\xi_1}^{\xi_2} (\xi_2 - \sigma)^{\rho_2-1} F(\sigma, H(\sigma)) d\sigma \right| \leq \frac{1}{\Gamma(\rho_2)} (\xi_2 - \xi_1) \|F\|,$$

by using

$$|F(\sigma, H(\sigma))| \leq \|F\|, \text{ and the factor } 0 \leq (\xi_2 - \sigma)^{\rho_2-1} \leq 1.$$

Let

$$\theta_s(\xi) = w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} I^{\rho_2},$$

for all  $s \geq 1$ . Then,

$$\|\theta_s\| \leq \|w_1\|^s \|I^{s(\rho_2+\rho_1)}\| \leq \|w_1\|^s \frac{1}{\Gamma((\rho_2 + \rho_1)s + 1)}.$$

Hence,

$$\begin{aligned} & \left| \sum_{s=1}^{\infty} (-1)^s \left[ I_{\xi=\xi_2}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} I^{\rho_2} F(\xi, H(\xi)) \right. \right. \\ & \quad \left. \left. - I_{\xi=\xi_1}^{\rho_2} w_1(\xi) I^{\rho_1} (I^{\rho_2} w_1(\xi) I^{\rho_1})^{s-1} I^{\rho_2} F(\xi, H(\xi)) \right] \right| \\ & \leq \frac{\xi_2 - \xi_1}{\Gamma(\rho_2)} \|F\| \sum_{s=1}^{\infty} \|w\|^s \frac{1}{\Gamma((\rho_2 + \rho_1)s + 1)} \leq \frac{\xi_2 - \xi_1}{\Gamma(\rho_2)} E_{\rho_2+\rho_1,1}(\|w_1\|) \|F\|, \end{aligned}$$

by the proof above.

In summary,

$$|T_1| \leq \frac{\xi_2 - \xi_1}{\Gamma(\rho_2)} \|F\| (E_{\rho_2+\rho_1,1}(\|w_1\|) + 1), \quad \text{noting that } \xi_2 - \xi_1 > 0,$$

which claims that  $T_1$  is a Lipschitz function. Similarly, we can show all other  $T_i$  for  $i = 2, \dots, 7$  are also Lipschitz functions. Thus,  $\mathbb{M}H$  is equicontinuous over  $[0, 1]$  for  $H \in \mathcal{B}$ . This implies  $\mathbb{M}$  is a compact operator using the Arzela–Ascoli theorem.

(iv) Lastly, one needs to prove the set

$$\{H \in C[0, 1] : H = \theta \mathbb{M}H \text{ for some } 0 < \theta \leq 1\}$$

is bounded. It holds since

$$\begin{aligned} \|H\| & \leq \frac{\|F\|}{Q_0} E_{\rho_2+\rho_1, \rho_2+1}(\|w_1\|) + \frac{1}{Q_0} \sum_{j=1}^l \|f_j\| E_{\rho_2+\rho_1, \gamma_j+\rho_2+1}(\|w_1\|) \\ & \quad + \frac{|w_3(0)|}{Q_0} E_{\rho_2+\rho_1, 1}(\|w_1\|) \\ & \quad + \frac{1}{Q_0} E_{\rho_2+\rho_1, 1}(\|w_1\|) \sum_{j=1}^l \frac{\|f_j\|}{\Gamma(\gamma_j + \rho_2 + 1)} + \frac{\|F\|}{Q_0 \Gamma(\rho_2 + 1)} E_{\rho_2+\rho_1, 1}(\|w_1\|) \end{aligned}$$

is uniformly bounded from the proof of Theorem 2 and by using

$$|w_3(H)| = |w_3(H) - w_3(0) + w_3(0)| \leq C \|H\| + |w_3(0)|$$

again, where

$$Q_0 = 1 - \left( \frac{\|w_1\|}{\Gamma(\rho_2 + 1)\Gamma(\rho_1 + 1)} + \|w_2\| + C \right) E_{\rho_2+\rho_1, 1}(\|w_1\|) > 0.$$

We finish the proof.  $\square$

**Remark 1.** (a) The conditions that  $w_3$  is a Lipschitz function and  $Q_0 > 0$  in Theorem 4 are essential to prove the existence results as they are required in the Leray–Schauder fixed-point theorem.

(b) Theorem 3 requires that all  $F$ ,  $f_j$ , and  $w_3$  are Lipschitz functions, while  $w_3$  is only a Lipschitz function in Theorem 4. In addition,  $A < 1$  in Theorem 3 implies  $Q_0 > 0$  in Theorem 4. But, the converse is not true, obviously.

#### 4. An Application

To complete this paper, we are going to provide a series solution to the following important Caputo fractional convection equation in  $\mathbb{R}^n$  for  $0 < \rho \leq 1$  to demonstrate applications of inverse operators in PDEs:

$$\begin{cases} {}_C D_t^\rho Y(t, x) + \sum_{s=1}^n \lambda_s \frac{\partial}{\partial x_s} Y(t, x) = v(t, x), & (t, x) \in [0, 1] \times [0, 1]^n, \\ Y(0, x) = Y_0(x), \end{cases} \quad (10)$$

where  $Y_0 \in C([0, 1]^n)$ ,  $\lambda_s \in \mathbb{R}$  for  $s = 1, \dots, n$  and

$${}_C D_t^\rho Y(t, x) = \frac{1}{\Gamma(1-\rho)} \int_0^t (t-\sigma)^{-\rho} Y'_t(\sigma, x) d\sigma.$$

This equation with the initial condition is vital in applications based on the following factors:

(a)  ${}_C D_t^\rho$  for  $0 < \rho \leq 1$  introduces memory effects into the evolution of the system. This is crucial for accurately modeling anomalous transport, subdiffusion, and history-dependent processes. Classical convection equations (with integer derivatives) assume that the system's rate of change depends only on the present, while this model assumes dependence on the entire history of the system.

(b) When  $\rho = 1$ , the equation reduces to the classical convection (or transport) equation:

$$Y_t + \sum_{s=1}^n \lambda_s \frac{\partial}{\partial x_s} Y(t, x) = v(t, x)$$

which is widely used in fluid mechanics, traffic flow, and wave propagation.

(c) For  $0 < \rho < 1$ , the equation captures subdiffusive dynamics, making it suitable for systems where transport is slower than classical models predict, such as in porous media, biological tissues, or financial markets.

(d) The term

$$\sum_{s=1}^n \lambda_s \frac{\partial}{\partial x_s} Y(t, x)$$

represents convection or directional transport, which appears in heat transfer, fluid flow, and atmospheric dynamics.

To begin our process, we define the partial fractional integral  $I_t^\rho$  of order  $\rho \in \mathbb{R}^+$  as

$$I_t^\rho Y(t, x) = \frac{1}{\Gamma(\rho)} \int_0^t (t-\sigma)^{\rho-1} Y(\sigma, x) d\sigma.$$

Then, apply  $I_t^\rho$  to Equation (10) to obtain

$$Y(t, x) + \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s} Y(t, x) = I_t^\rho v(t, x) + Y_0(x),$$

using

$$\begin{aligned} Y(0, x) &= Y_0(x), \text{ and} \\ (I_t^\rho {}_C D_t^\rho) Y(t, x) &= Y(t, x) - Y(0, x) = Y(t, x) - Y_0(x). \end{aligned}$$

Thus,

$$\left(1 + \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s}\right) Y(t, x) = I_t^\rho v(t, x) + Y_0(x). \tag{11}$$

We first claim that the inverse operator of  $1 + \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s}$  is

$$\begin{aligned} \mathcal{V} &= \sum_{s_1=0}^{\infty} (-1)^{s_1} \left( \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s} \right)^{s_1} \\ &= \sum_{s_1=0}^{\infty} (-1)^{s_1} I_t^{\rho s_1} \sum_{j_1+\dots+j_n=s_1} \binom{s_1}{j_1, \dots, j_n} \lambda_1^{j_1} \dots \lambda_n^{j_n} \frac{\partial^{j_1+\dots+j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} \end{aligned}$$

in the subspace  $S$  of  $C([0, 1] \times [0, 1]^n)$ , given by

$$\begin{aligned} S &= \{Y(t, x) \in C([0, 1] \times [0, 1]^n) : \exists \text{ a constant } M_Y > 0 \text{ such that} \\ &\left. \left| \frac{\partial^{j_1+\dots+j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} Y(t, x) \right| \leq M_Y^{j_1+\dots+j_n} \right\}. \end{aligned}$$

Clearly,

$$\sum_{j_1+\dots+j_n=s_1} \binom{s_1}{j_1, \dots, j_n} = n^{s_1}.$$

Letting

$$\lambda = \max\{|\lambda_1|, \dots, |\lambda_n|\},$$

we have, for any  $Y \in S$ ,

$$\|\mathcal{V}Y\| \leq \sum_{s_1=0}^{\infty} \|I_t^{\rho s_1}\| n^{s_1} \lambda^{s_1} M_Y^{s_1} \leq \sum_{s_1=0}^{\infty} \frac{(n\lambda M_Y)^{s_1}}{\Gamma(\rho s_1 + 1)} = E_{\rho,1}(n\lambda M_Y) < +\infty,$$

which implies that  $\mathcal{V}$  is well-defined.

In addition,  $\mathcal{V}$  is an inverse operator since

$$\mathcal{V} \left(1 + \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s}\right) = \left(1 + \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s}\right) \mathcal{V} = 1.$$

In fact,

$$\begin{aligned} \mathcal{V} \left(1 + \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s}\right) &= \mathcal{V} + \sum_{s_1=0}^{\infty} (-1)^{s_1} \left( \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s} \right)^{s_1+1} \\ &= 1 + \sum_{s_1=1}^{\infty} (-1)^{s_1} \left( \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s} \right)^{s_1} + \sum_{s_1=0}^{\infty} (-1)^{s_1} \left( \sum_{s=1}^n \lambda_s I_t^\rho \frac{\partial}{\partial x_s} \right)^{s_1+1} \\ &= 1, \end{aligned}$$

and  $\mathcal{V}$  is unique.

From Equation (11), we find a series solution to Equation (10) in  $C([0, 1] \times [0, 1]^n)$  as

$$\begin{aligned}
 Y(t, x) &= \sum_{s=0}^{\infty} (-1)^s I_t^{\rho s + \rho} \sum_{j_1 + \dots + j_n = s} \binom{s}{j_1, \dots, j_n} \lambda_1^{j_1} \dots \lambda_n^{j_n} \frac{\partial^{j_1 + \dots + j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} v(t, x) \\
 &\quad + \sum_{s=0}^{\infty} (-1)^s I_t^{\rho s} \sum_{j_1 + \dots + j_n = s} \binom{s}{j_1, \dots, j_n} \lambda_1^{j_1} \dots \lambda_n^{j_n} \frac{\partial^{j_1 + \dots + j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} Y_0(x) \\
 &= \sum_{s=0}^{\infty} (-1)^s I_t^{\rho s + \rho} \sum_{j_1 + \dots + j_n = s} \binom{s}{j_1, \dots, j_n} \lambda_1^{j_1} \dots \lambda_n^{j_n} \frac{\partial^{j_1 + \dots + j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} v(t, x) \\
 &\quad + \sum_{s=0}^{\infty} (-1)^s \frac{t^{\rho s}}{\Gamma(\rho s + 1)} \sum_{j_1 + \dots + j_n = k} \binom{s}{j_1, \dots, j_n} \lambda_1^{j_1} \dots \lambda_n^{j_n} \frac{\partial^{j_1 + \dots + j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} Y_0(x),
 \end{aligned}$$

where  $v, Y_0 \in S$ .

Clearly, the solution  $Y(t, x)$  given above is convergent in the space  $C([0, 1] \times [0, 1]^n)$ . Indeed,

$$\begin{aligned}
 \|Y\| &\leq \sum_{s=0}^{\infty} \frac{1}{\Gamma(\rho s + \rho + 1)} \sum_{j_1 + \dots + j_n = s} \binom{s}{j_1, \dots, j_n} |\lambda_1|^{j_1} \dots |\lambda_n|^{j_n} M_v^{j_1 + \dots + j_n} \\
 &\quad + \sum_{s=0}^{\infty} \frac{1}{\Gamma(\rho s + 1)} \sum_{j_1 + \dots + j_n = s} \binom{s}{j_1, \dots, j_n} |\lambda_1|^{j_1} \dots |\lambda_n|^{j_n} M_{Y_0}^{j_1 + \dots + j_n} \\
 &= \sum_{s=0}^{\infty} \frac{n^s \lambda^s M_v^s}{\Gamma(\rho s + \rho + 1)} + \sum_{s=0}^{\infty} \frac{n^s \lambda^s M_{Y_0}^s}{\Gamma(\rho s + 1)} = E_{\rho, \rho + 1}(n\lambda M_v) + E_{\rho, 1}(n\lambda M_{Y_0}) < +\infty.
 \end{aligned}$$

In particular, for  $Y_0(x) = 0$ , we have

$$Y(t, x) = \sum_{s=0}^{\infty} (-1)^s I_t^{\rho s + \rho} \sum_{j_1 + \dots + j_n = s} \binom{s}{j_1, \dots, j_n} \lambda_1^{j_1} \dots \lambda_n^{j_n} \frac{\partial^{j_1 + \dots + j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} v(t, x).$$

Evidently, if  $\lambda_1 = \lambda_2 = \dots = \lambda_n$ , then Equation (10) is

$$\begin{cases} {}_C D_t^\rho Y(t, x) + \lambda_1 \nabla Y(t, x) = v(t, x), & (t, x) \in [0, 1] \times [0, 1]^n, \\ Y(0, x) = Y_0(x). \end{cases}$$

**Example 3.** The equation for  $0 < \rho \leq 1$ :

$$\begin{cases} {}_C D_t^\rho Y(t, x) + \frac{\partial}{\partial x} Y(t, x) = e^{xt}, & (t, x) \in [0, 1] \times [0, 1], \\ Y(0, x) = 0, \end{cases} \tag{12}$$

has a series solution

$$Y(t, x) = \sum_{s=0}^{\infty} (-1)^s \sum_{r=0}^{\infty} \frac{x^r \Gamma(s + r + 1) t^{(\rho + 1)s + \rho + r}}{r! \Gamma((\rho + 1)s + \rho + r + 1)}$$

in  $C[0, 1] \times [0, 1]$ .

**Proof.** Clearly,  $e^{xt} \in S$ , and

$$\begin{aligned} Y(t, x) &= \sum_{s=0}^{\infty} (-1)^s I_t^{\rho s + \rho} \sum_{j_1 + \dots + j_n = s} \binom{s}{j_1, \dots, j_n} \lambda_1^{j_1} \dots \lambda_n^{j_n} \frac{\partial^{j_1 + \dots + j_n}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} v(t, x) \\ &= \sum_{s=0}^{\infty} (-1)^s \sum_{r=0}^{\infty} \frac{x^r I_t^{\rho s + \rho} t^{s+r}}{r!} \\ &= \sum_{s=0}^{\infty} (-1)^s \sum_{r=0}^{\infty} \frac{x^r \Gamma(s+r+1) t^{(\rho+1)s + \rho + r}}{r! \Gamma((\rho+1)s + \rho + r + 1)}. \end{aligned}$$

□

**Remark 2.** There are many investigations on numerical solutions to various time-fractional convection equations; see [14], for example. They are different from our inverse operator techniques and only find approximate solutions on finite domains in general, with or without convergence analysis. Solutions derived here are exact and well-defined in the space  $C([0, 1] \times [0, 1]^n)$ . However, it seems difficult and challenging to consider boundary value problems. In particular, if all  $\lambda_j = 0$ , then

$$Y(t, x) = I_t^\rho v(t, x) + Y_0(x)$$

clearly is the solution satisfying the initial condition.

In addition, the inverse operator method can be broadly applied to the study of various fractional partial differential equations, such as the generalized multi-term time-fractional diffusion-wave and partial integro-differential equation [4], the fractional convection–diffusion equation, and the generalized wave equation [15].

Recently, Li and Wang [14] applied the non-uniform L1/discontinuous Galerkin (DG) finite element method to study numerical solutions for the following two-dimensional problem for  $0 < \alpha < 1$ :

$$\begin{cases} {}_C D_t^\alpha \chi(t, x, y) + \frac{\partial}{\partial x_1} \gamma_1(x, y) \chi(t, x, y) + \frac{\partial}{\partial x_2} \gamma_2(x, y) \chi(t, x, y) \\ = f(t, x, y), \quad (t, x, y) \in (0, T] \times \Omega, \\ \chi(0, x, y) = \phi(x, y), \quad (x, y) \in \Omega = (a_1, b_1) \times (a_2, b_2) \quad (\text{finite domain}), \\ \chi(t, a_1, y) = \chi(t, x, a_2) = 0, \quad (t, x, y) \in (0, T] \times \partial\Omega. \end{cases}$$

## 5. Conclusions

We have investigated the uniqueness, existence, and stability of the new Equation (1) using a functional inverse operator, the Mittag–Leffler function, and several fixed-point theorems in the Banach space  $C[0, 1]$ . Multiple examples were presented to illustrate the applicability of the results. In addition, we applied the inverse operator method to solve the Caputo fractional convection Equation (10) in  $\mathbb{R}^n$  by introducing a newly constructed function space  $S$ . This represents a novel approach for analyzing a wide class of well-known partial differential equations—such as time-fractional diffusion equations involving the Laplacian operator  $\Delta$ —through the framework of inverse operator techniques.

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