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EXISTENCE FOR A NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WITH HILFER FRACTIONAL DERIVATIVE

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We study the uniqueness of solutions to a new nonlinear Hilfer integro-differential equation with an initial condition and arbitrary numbers of the Riemann–Liouville fractional integral operators. Our investigation is based on an equivalent implicit integral equation in series obtained from Babenko's approach, the multivariate Mittag-Leffler function as well as Banach's contractive principle in a new Banach space. The technique used clearly opens up new directions for studying other types of initial or boundary value problems with different fractional derivatives and variable coefficients. An illustrative example is provided to demonstrate applications of the key theorem.

1. Introduction

Let $-\infty < a < b < +\infty$ and $\lambda_i \in R$ for $i = 1, 2, \dots, m$. We shall consider the following nonlinear integro-differential equation with an initial condition:

$$\begin{cases} D_{a+}^{\alpha,\beta} u(x) + \sum_{i=1}^{m} \lambda_i \, I_{a+}^{\beta_i} u(x) = I_{a+}^{\beta} g(x, u(x)), \; 0 < \alpha < 1, \; 0 \le \beta < 1, \; \beta_i \ge \beta, \\ I_{a+}^{1-\gamma} u(a) = u_a \in R, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1-1}$$

where $x \in (a, b]$ and $D_{a+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β [8; 13], which is an interpolation between the Riemann–Liouville and Caputo fractional derivatives. The operator $I_{a+}^{\beta_i}$ is the Riemann–Liouville fractional integral of the order β_i, the nonlinear term $g : (a, b] \times R \to R$ is a function satisfying certain conditions. In 2000, Hilfer introduced the Hilfer fractional derivative which combines Caputo and Riemann–Liouville fractional derivatives, and can be used in the theoretical simulation of dielectric relaxation in glass forming materials [6; 7]. Sandev et al. [14] derived the existence results of the fractional diffusion equation with the Hilfer fractional derivative which attained in terms of the Mittag-Leffler functions. In 2015, Gu and Trujillo [4] studied the existence results of the fractional differential equations with the Hilfer derivative based on noncompact measure method.

Clearly, the parameter γ satisfies

$$0 < \max\{\alpha, \beta\} \le \gamma < 1, \quad 1-\gamma < 1 - \beta(1-\alpha).$$

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The two-parameter fractional derivative $D_{a+}^{\alpha,\beta}$ generates more types of stationary states and gives an extra degree of freedom on the initial condition with applications in physics [6; 7; 3]. In 2012, Furati et al. [2] studied the following nonlinear Hilfer differential equation with an initial condition:

$$\begin{cases} D_{a+}^{\alpha,\beta}u(x) = g(x, u(x)), \ 0 < \alpha < 1, \ 0 \le \beta \le 1, x \in (a, b], \\ I_{a+}^{1-\gamma}u(a^+) = u_a \in R, \quad \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{1-2}$$

They proved the existence and uniqueness of global solutions in a space of weighted continuous functions using Banach's fixed point theorem. More generally, Wang and Zhang [15] considered the existence of solutions to the following nonlocal initial value problem in 2015:

$$\begin{cases} D_{a+}^{\alpha,\beta}u(x) = g(x, u(x)), \ 0 < \alpha < 1, \ 0 \le \beta \le 1, x \in (a, b], \\ I_{a+}^{1-\gamma}u(a^+) = \sum_{i=1}^{m} \lambda_i u(\tau_i), \quad \gamma = \alpha + \beta - \alpha\beta, \quad \tau_i \in (a, b]. \end{cases} \tag{1-3}$$

Outline. Section 2 introduces some basic concepts, a Banach space $C_{1-\gamma}[a, b]$ with $\gamma = \alpha+\beta-\alpha\beta$, $\beta < 1$, a subspace $W_\gamma[a, b] \subset C_{1-\gamma}[a, b]$, the multivariate Mittag-Leffler function and Babenko's approach. In addition, we convert (1-1) to an equivalent implicit integral equation in series using Babenko's technique. Then we obtain sufficient conditions for the uniqueness of solutions with the help of Banach's contractive principle in the Banach space $W_\gamma[a, b]$, and further demonstrate applications of the main result by an example in Section 3.

2. Preliminaries

The Riemann–Liouville fractional integral of the order $s \ge 0$ of function $u(x)$ is defined by [12]

$$(I_{a+}^s u)(x) = \frac{1}{\Gamma(s)} \int_a^x (x-t)^{s-1} u(t)dt, \quad x > a,$$

and

$$I_{a+}^0 u(x) = u(x),$$

from [10].

The Riemann–Liouville fractional derivative of order $\alpha \in [n-1, n)$, for $n \in N$, of function $u(x)$ is defined by [13]

$$\left(D_{a+}^\alpha u\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} u(t)dt, \quad x > a.$$

The Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \le \beta \le 1$ of a function $u(x)$ is defined by [6]

$$D_{a+}^{\alpha,\beta}u(x) = (I_{a+}^{\beta(1-\alpha)} D I_{a+}^{(1-\beta)(1-\alpha)})u(x),$$

where $D = d/dx$.

It follows from [6] that the operator $D_{a+}^{\alpha,\beta}$ can also be written as $D_{a+}^{\alpha,\beta} = I_{a+}^{\beta(1-\alpha)} D_{a+}^{\gamma}$, where $\alpha \le \gamma = \alpha + \beta - \alpha\beta$. The Riemann–Liouville fractional derivative is the case $D_a^\alpha = D_{a+}^{\alpha,0}$ and the Caputo fractional derivative is the case ${}^C D_a^\alpha = D_{a+}^{\alpha,1}$.

For any $0 \le \gamma < 1$, we define the Banach space $C_{1-\gamma}[a,b]$ as

$$C_{1-\gamma}[a,b] = \{u : (a,b] \to R : (x-a)^{1-\gamma} u(x) \in C[a,b]\},$$

with the norm

$$\|u\|_{C_{1-\gamma}} = \max_{x \in [a,b]} |(x-a)^{1-\gamma} u(x)|.$$

Clearly, $C[a,b] \subset C_{1-\gamma}[a,b]$ for any $0 \le \gamma < 1$. A subspace $C^1_{1-\gamma}[a,b] \subset C_{1-\gamma}[a,b]$ is defined by

$$C^1_{1-\gamma}[a,b] = \{u \in C[a,b] : u' \in C_{1-\gamma}[a,b]\},$$

with the norm

$$\|u\|_{C^1_{1-\gamma}} = \|u\|_C + \|u'\|_{C_{1-\gamma}}.$$

Evidently, $C^1_{1-\gamma}[a,b]$ is a Banach space. Finally, the Banach space $W_\gamma[a,b]$ is defined as

$$W_\gamma[a,b] = \{u \in C_{1-\gamma}[a,b] : I^{1-\gamma}_{a+} u \in C^1_{1-\gamma}[a,b]\} \subset C_{1-\gamma}[a,b],$$

with the norm

$$\|u\|_{W_\gamma} = \max\{\|u\|_{C_{1-\gamma}}, \|I^{1-\gamma}_{a+} u\|_C, \|D I^{1-\gamma}_{a+} u\|_{C_{1-\gamma}}\}.$$

Lemma 1 (see [2]). *Let $0 < \alpha < 1$ and $\gamma = \alpha + \beta - \alpha\beta$ with $0 \le \beta < 1$. If $u \in C_{1-\gamma}[a,b]$ and $I^{1-\beta+\alpha\beta}_{a+} u \in C^1_{1-\gamma}[a,b]$, then $D^{\alpha,\beta}_{a+} I^{\alpha}_{a+} u$ exists in $(a,b]$ and*

$$D^{\alpha,\beta}_{a+} I^{\alpha}_{a+} u = u,$$

for all $x \in (a,b]$.

Lemma 2 [13]. *Let $0 < t < 1$ and $0 \le s < 1$. If $u \in C_s[a,b]$ and $I^{1-t}_{a+} u \in C^1_s[a,b]$, then*

$$I^t_{a+} D^t_{a+} u(x) = u(x) - \frac{I^{1-t}_{a+} u(a)}{\Gamma(t)} (x-a)^{t-1},$$

for all $x \in (a,b]$.

It follows from $t = \gamma$ and $s = 1-\gamma$ that

$$I^\gamma_{a+} D^\gamma_{a+} u(x) = u(x) - \frac{I^{1-\gamma}_{a+} u(a)}{\Gamma(\gamma)} (x-a)^{\gamma-1},$$

for all $x \in (a,b]$ and $u \in W_\gamma[a,b]$.

The multivariate Mittag-Leffler function [5] is defined by

$$E_{(\alpha_1,\dots,\alpha_m),\beta}(z_1,\dots,z_m) = \sum_{k=0}^{\infty} \sum_{k_1+\dots+k_m=k} \binom{k}{k_1,\dots,k_m} \frac{z_1^{k_1} \cdots z_m^{k_m}}{\Gamma(\alpha_1 k_1 + \dots + \alpha_m k_m + \beta)},$$

where $\alpha_i, \beta > 0$ and $z_i \in \mathbb{C}$ for $i = 1,2,\dots,m$ and

$$\binom{k}{k_1,\dots,k_m} = \frac{k!}{k_1! \cdots k_m!}.$$

3. Main results

Babenko's approach [1] is a powerful method for solving differential equations with initial conditions, as well as integral equations. It is generally the same as the Laplace transform when dealing with equations with constant coefficients, but it can also be applied to differential and integral equations with continuous variable coefficients and boundary value problems [9; 11]. To show the applications of this approach, we will deduce an implicit integral equation which is equivalent to (1-1) in the space $W_\gamma[a,b]$.

Theorem 3. *Let $g:(a,b]\times R\to R$ be a continuous and bounded function. Then* (1-1) *is equivalent to the following implicit integral equation in the space $W_\gamma[a,b]$.*

(3-1)

$$u(x)=\frac{u_a}{\Gamma(\gamma)}\sum_{k=0}^{\infty}(-1)^k\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m}\,I_{a+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(x-a)^{\gamma-1}$$
$$+\sum_{k=0}^{\infty}(-1)^k\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m}\,I_{a+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}g(x,u(x)),$$

Proof. Clearly for $\gamma=\alpha+\beta-\alpha\beta$ with $\beta<1$,

$$I_{a+}^{\alpha}D_{a+}^{\alpha,\beta}u(x)=I_{a+}^{\alpha}I_{a+}^{\beta(1-\alpha)}D_{a+}^{\gamma}u(x)=I_{a+}^{\gamma}D_{a+}^{\gamma}u(x)=u(x)-\frac{I_{a+}^{1-\gamma}u(a)}{\Gamma(\gamma)}(x-a)^{\gamma-1},$$

for all $x\in(a,b]$ and $u\in W_\gamma[a,b]$.

Applying the operator I_{a+}^{α} to both sides of the equation $D_{a+}^{\alpha,\beta}u(x)+\sum_{i=1}^{m}\lambda_i\,I_{a+}^{\beta_i}u(x)=I_{a+}^{\beta}g(x,u(x))$, we come to

$$u(x)-\frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+\sum_{i=1}^{m}\lambda_i\,I_{a+}^{\alpha+\beta_i}u(x)=I_{a+}^{\alpha+\beta}g(x,u(x)),$$

using the initial condition $I_{a+}^{1-\gamma}u(a)=u_a$. This implies that

$$\Big(1+\sum_{i=1}^{m}\lambda_i\,I_{a+}^{\alpha+\beta_i}\Big)u(x)=\frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+I_{a+}^{\alpha+\beta}g(x,u(x)).$$

Treating the factor in front of $u(x)$ as a variable, we derive by Babenko's approach that

$$\begin{aligned}
u(x)&=\Big(1+\sum_{i=1}^{m}\lambda_i\,I_{a+}^{\alpha+\beta_i}\Big)^{-1}\Big(\frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+I_{a+}^{\alpha+\beta}g(x,u(x))\Big)\\
&=\sum_{k=0}^{\infty}(-1)^k\Big(\sum_{i=1}^{m}\lambda_i\,I_{a+}^{\alpha+\beta_i}\Big)^k\Big(\frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+I_{a+}^{\alpha+\beta}g(x,u(x))\Big)\\
&=\sum_{k=0}^{\infty}(-1)^k\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}I_{a+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}\\
&\qquad\qquad\times\Big(\frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+I_{a+}^{\alpha+\beta}g(x,u(x))\Big)\\
&=\frac{u_a}{\Gamma(\gamma)}\sum_{k=0}^{\infty}(-1)^k\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m}\\
&\qquad\qquad\times I_{a+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(x-a)^{\gamma-1}
\end{aligned}$$

$$+\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m} \times I_{a^+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} g(x,u(x)).$$

Next, we show that $u \in W_\gamma[a,b]$. Clearly, $I_{a^+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(x-a)^{\gamma-1}$ equals

$$\frac{\Gamma(\gamma)}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma)}(x-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma-1}$$

and $\max_{x\in[a,b]}|(x-a)^{1-\gamma}u(x)|$ is bounded above by

$$\begin{aligned}
&|u_a|\sum_{k=0}^{\infty}\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
&\qquad\times\frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma)} \\
&\quad+(b-a)^{1+\alpha\beta}\sum_{k=0}^{\infty}\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
&\qquad\times\frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\alpha+\beta+1)}\sup_{x\in(a,b]}|g(x,u(x))| \\
&=|u_a|E_{(\alpha+\beta_1,\ldots,\alpha+\beta_m),\gamma}\big(|\lambda_1|(b-a)^{\alpha+\beta_1},\ldots,|\lambda_m|(b-a)^{\alpha+\beta_m}\big) \\
&\quad+(b-a)^{1+\alpha\beta}E_{(\alpha+\beta_1,\ldots,\alpha+\beta_m),\alpha+\beta+1}\big(|\lambda_1|(b-a)^{\alpha+\beta_1},\ldots,|\lambda_m|(b-a)^{\alpha+\beta_m}\big) \\
&\qquad\times\sup_{x\in(a,b]}|g(x,u(x))|<+\infty.
\end{aligned}$$

Since $I_{a^+}^{1-\gamma}(x-a)^{\gamma-1}=\Gamma(\gamma)$, we get

$$\begin{aligned}
\|I_{a^+}^{1-\gamma}u\|_C &\le |u_a|\sum_{k=0}^{\infty}\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
&\qquad\times\frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+1)} \\
&\quad+\sum_{k=0}^{\infty}\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
&\qquad\times\frac{(b-a)^{1+\alpha\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+2+\alpha\beta)}\sup_{x\in(a,b]}|g(x,u(x))| \\
&=|u_a|E_{(\alpha+\beta_1,\ldots,\alpha+\beta_m),1}\big(|\lambda_1|(b-a)^{\alpha+\beta_1},\ldots,|\lambda_m|(b-a)^{\alpha+\beta_m}\big) \\
&\quad+(b-a)^{1+\alpha\beta}E_{(\alpha+\beta_1,\ldots,\alpha+\beta_m),2+\alpha\beta}\big(|\lambda_1|(b-a)^{\alpha+\beta_1},\ldots,|\lambda_m|(b-a)^{\alpha+\beta_m}\big) \\
&\qquad\times\sup_{x\in(a,b]}|g(x,u(x))|<+\infty.
\end{aligned}$$

Finally, we consider the norm

$$\begin{aligned}\|DI_{a+}^{1-\gamma}u\|_{C_{1-\gamma}} &= \max_{x\in[a,b]}\left|(x-a)^{1-\gamma}DI_{a+}^{1-\gamma}u\right| \\ &\leq |u_a|\sum_{k=1}^{\infty}\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\ &\qquad\times\frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m-\gamma}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m)} \\ &\quad+(b-a)^{1-\gamma+\alpha\beta}\sum_{k=0}^{\infty}\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\ &\qquad\times\frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\alpha\beta+1)}<+\infty,\end{aligned}$$

by noting that

$$(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m-\gamma\geq 0,$$

for $k_1+\cdots+k_m=k\geq 1$, since $\beta_i\geq\beta$ for all $i=1,2,\ldots,m$. In summary, $u\in W_\gamma[a,b]$.

Conversely, if u is given by (3-1) in the space $W_\gamma[a,b]$ then $I_{a+}^{1-\gamma}u(a)=u_a$. Indeed,

$$\begin{aligned}u(x) &= \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+\frac{u_a}{\Gamma(\gamma)}\sum_{k=1}^{\infty}(-1)^k\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m} \\ &\qquad\times I_{a+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(x-a)^{\gamma-1} \\ &\quad+\sum_{k=0}^{\infty}(-1)^k\sum_{k_1+k_2+\cdots+k_m=k}\binom{k}{k_1,k_2,\ldots,k_m}\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m} \\ &\qquad\times I_{a+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}g(x,u(x)) \\ &= \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+u_1(x).\end{aligned}$$

Using $I_{a+}^{1-\gamma}(x-a)^{\gamma-1}=\Gamma(\gamma)$, and noting that $I_{a+}^{1-\gamma}u_1(x)\big|_{x=a}=0$, we obtain $I_{a+}^{1-\gamma}u(a)=u_a$.

Applying the operator $D_{a+}^{\alpha,\beta}$ to

$$\Big(1+\sum_{i=1}^{m}\lambda_i\,I_{a+}^{\alpha+\beta_i}\Big)u(x)=\frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}+I_{a+}^{\alpha+\beta}g(x,u(x)),$$

which is equivalent to (3-1), we obtain

$$D_{a+}^{\alpha,\beta}u(x)+\sum_{i=1}^{m}\lambda_i\,D_{a+}^{\alpha,\beta}I_{a+}^{\alpha+\beta_i}u(x)=\frac{u_a}{\Gamma(\gamma)}D_{a+}^{\alpha,\beta}(x-a)^{\gamma-1}+D_{a+}^{\alpha,\beta}I_{a+}^{\alpha+\beta}g(x,u(x)).$$

Clearly,

$$D_{a+}^{\alpha,\beta}(x-a)^{\gamma-1}=I_{a+}^{\beta(1-\alpha)}(D_{a+}^{\gamma}(x-a)^{\gamma-1})=I_{a+}^{\beta(1-\alpha)}0=0$$

for $0<\gamma<1$, and

$$D_{a+}^{\alpha,\beta}I_{a+}^{\alpha+\beta_i}u=D_{a+}^{\alpha,\beta}I_{a+}^{\alpha}I_{a+}^{\beta_i}u=I_{a+}^{\beta_i}u,$$

by Lemma 1, since $I_{a+}^{\beta_i} u \in C_{1-\gamma}[a,b]$ and $I_{a+}^{1+\beta_i-\beta+\alpha\beta} u \in C_{1-\gamma}^1[a,b]$. Similarly,

$$D_{a+}^{\alpha,\beta} I_{a+}^{\alpha+\beta} g(x,u(x)) = I_{a+}^{\beta} g(x,u(x)).$$

Hence, u satisfies (1-1). This completes the proof of Theorem 3. □

Theorem 4. *Let $g:(a,b]\times R \to R$ be a continuous and bounded function satisfying*

$$|g(x,y_1)-g(x,y_2)| \le L|y_1-y_2|, \quad y_1,y_2 \in R,$$

for a nonnegative constant L. Furthermore, assume

$$q = L(b-a)^{\alpha+\beta}\Gamma(\gamma) E_{(\alpha+\beta_1,\dots,\alpha+\beta_m),\,2(\alpha+\beta)-\alpha\beta}\big(|\lambda_1|(b-a)^{\alpha+\beta_1},\dots,|\lambda_m|(b-a)^{\alpha+\beta_m}\big) < 1.$$

Then (1-1) *has a unique solution in the space $W_\gamma[a,b]$.*

Proof. For $u \in C_{1-\gamma}[a,b]$, we define a nonlinear mapping T over the space $C_{1-\gamma}[a,b]$ by

$$\begin{aligned}(Tu)(x) = \frac{u_a}{\Gamma(\gamma)} \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,k_2,\dots,k_m} \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m} \\ \times I_{a+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(x-a)^{\gamma-1} \\ + \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,k_2,\dots,k_m} \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m} \\ \times I_{a+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} g(x,u(x)).\end{aligned}$$

It follows from the proof of Theorem 3 that $Tu \in C_{1-\gamma}[a,b]$. We further show that T is contractive. In fact, we get for $u,v \in\in C_{1-\gamma}[a,b]$

$$\begin{aligned}\|Tu-Tv\|_{C_{1-\gamma}} &= \max_{x\in[a,b]} \Big| (x-a)^{1-\gamma} \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,k_2,\dots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \\ &\qquad \times I_{a+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(g(x,u)-g(x,v)) \Big| \\ &\le L(b-a)^{\alpha+\beta}\Gamma(\gamma) \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,k_2,\dots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\ &\qquad \times \frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma(2(\alpha+\beta)-\alpha\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m)} \|u-v\|_{C_{1-\gamma}} \\ &= q\|u-v\|_{C_{1-\gamma}},\end{aligned}$$

using

$$g(x,u)-g(x,v) = (x-a)^{\gamma-1}[(x-a)^{1-\gamma}(g(x,u)-g(x,v))],$$

and

$$I_{a+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(x-a)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m)} \times (x-a)^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma-1}$$
$$= \frac{\Gamma(\gamma)}{\Gamma(2(\alpha+\beta)-\alpha\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m)} \times (x-a)^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma-1}.$$

Since $q < 1$, the mapping T has a unique fixed point in the space $C_{1-\gamma}[a,b]$. If $u \in W_\gamma[a,b]$ then $Tu \in W_\gamma[a,b]$. Therefore, T has a unique fixed point in $W_\gamma[a,b]$ in the sense of the topology in $C_{1-\gamma}[a,b]$. This implies that (1-1) has a unique solution in $W_\gamma[a,b]$, proving Theorem 4. □

Example 5. The following nonlinear Hilfer integro-differential equation with an initial condition has a unique solution in the space $W_{0.76}[0,1]$. given by

(3-2)
$$\begin{cases} D_{0+}^{0.6,0.4}u(x) + 2\, I_{0+}^{0.4}u(x) - 2\, I_{0+}^{0.5}u(x) = \frac{1}{23} I_{0+}^{0.4} \sin u(x), \\ I_{0+}^{0.24}u(0) = -4. \end{cases}$$

Proof. Since $g(x,u(x)) = \frac{1}{23}\sin u(x)$, we have $|g(x,u(x)) - g(x,v(x))| \le \frac{1}{23}|u(x)-v(x)|$ and

$$q = \frac{1}{23}\Gamma(0.76) E_{(1,1.1),\,1.76}(2,2) = \frac{1}{23}\Gamma(0.76) \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} \frac{2^{k_1}2^{k_2}}{\Gamma(k_1+1.1k_2+1.76)}.$$

Applying

$$\sum_{k_1+k_2=k} \binom{k}{k_1,k_2} = 2^k$$

and

$$\frac{2^{k_1}2^{k_2}}{\Gamma(k_1+1.1k_2+1.76)} \le \frac{2^k}{\Gamma(k+1.76)},$$

we conclude that

$$q \le \frac{1}{23}\Gamma(0.76) \sum_{k=0}^{\infty} \frac{4^k}{\Gamma(k+1.76)} \approx \frac{22.8418}{23} < 1,$$

since

(3-3)
$$\Gamma(0.76) \sum_{k=0}^{\infty} \frac{4^k}{\Gamma(k+1.76)} \approx 22.8418,$$

From Theorem 4, (3-3) has a unique solution in the space $W_{0.76}[0,1]$. This completes the proof. □

Remark 6. From the proof of Theorem 3, we deduce that

$$u(x) = \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} - \sum_{i=1}^{m} \lambda_i\, I_{a+}^{\alpha+\beta_i} u(x) + I_{a+}^{\alpha+\beta} g(x,u(x)),$$

which can be used in finding an approximate solution by the following recursion:

$$u_n(x) = \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} - \sum_{i=1}^{m} \lambda_i \, I_{a+}^{\alpha+\beta_i} u_{n-1}(x) + I_{a+}^{\alpha+\beta} g(x, u_{n-1}(x)),$$

for $n = 1, 2, \ldots$, and an initial function $u_0(x)$.

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