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Article

The Time–Fractional Wave Equation with Variable Coefficients

Chenkuan Li

Department of Mathematics and Computer Science, Brandon University, Brandon, MB R7A 6A9, Canada; lic@brandonu.ca

Abstract

In this paper, we primarily use the inverse operator method to find a unique series solution to a time–fractional wave equation with variable coefficients based on the Mittag–Leffler function. In addition, we also derive the series and integral convolution solutions to the Klein–Gordon equation using the Fourier transform and Green's functions. Furthermore, our series solutions significantly simplify the process of finding solutions with several illustrative examples, avoiding the need for complicated integral computations.

Keywords: Mittag–Leffler function; time–fractional wave equation; inverse operator method; Fourier transform; time–fractional Klein–Gordon equation

MSC: 35A02; 35C15; 47N20; 26A33

1. Introduction

The induction will proceed through six small subsections: preliminaries, motivation, a review of the literature, the Klein–Gordon equation and research methods, the importance of the Klein–Gordon equation, and further discussion and examples.

1.1. Preliminaries

Time–fractional wave equations with variable coefficients play a significant role in both theoretical and applied mathematics, as well as in physics and engineering, due to their ability to model complex phenomena that classical wave equations cannot adequately capture. Moreover, the Fourier transform serves as a fundamental tool in the analysis of such equations. To initiate this process, we define the n-dimensional Fourier transform as

$$\mathcal{F}\{\psi\}(\zeta) = \tilde{\psi}(\zeta) = \int_{\mathbb{R}^n} \psi(x) e^{-i\langle \zeta, x\rangle} dx, \quad \zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{R}^n,$$

and the inverse Fourier transform;

$$\psi(x) = \mathcal{F}^{-1}\{\tilde{\psi}\}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\psi}(\zeta) e^{i\langle \zeta, x\rangle} d\zeta,$$

where $\langle \zeta, x\rangle := \zeta_1 x_1 + \cdots + \zeta_n x_n$. In particular, we write $|\zeta|^2 = \zeta_1^2 + \cdots + \zeta_n^2$.

Clearly, from $\mathcal{F}\{\triangle\psi\}(\zeta) = -|\zeta|^2 \tilde{\psi}(\zeta)$, we have

$$\triangle^j \psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-|\zeta|^2)^j \tilde{\psi}(\zeta) e^{i\langle \zeta, x\rangle} d\zeta,$$

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where $\triangle$ is the Laplace operator given by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

Let $\alpha_1, \cdots, \alpha_m, \beta > 0$ and $z_1, \cdots, z_m \in \mathbb{C}$. Then

$$\begin{aligned} & E_{(\alpha_1,\cdots,\alpha_m),\,\beta}(z_1, \cdots, z_m) \\ & = \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_m=j} \binom{j}{j_1, \cdots, j_m} \frac{z_1^{j_1} \cdots z_m^{j_m}}{\Gamma(\alpha_1 j_1 + \cdots + \alpha_m j_m + \beta)} \end{aligned}$$

is the well-known multivariate Mittag–Leffler function [1,2], which is an entire function on the complex plane $\mathbb{C}^m$, with

$$\binom{j}{j_1, \cdots, j_m} = \frac{j!}{j_1! \cdots j_m!}.$$

When $m = 1$, it reduces to the following two-parameter Mittag–Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C}.$$

If $\beta = 1$, we obtain the classical Mittag–Leffler function defined by

$$E_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad \alpha > 0,\ z \in \mathbb{C}.$$

The gamma function, denoted by $\Gamma(z)$ for $z \in \mathbb{C}$, is an extension of the factorial function, given by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \mathrm{Re}(z) > 0.$$

The partial Liouville–Caputo fractional derivative ${}_c\partial^\alpha/\partial t^\alpha$ of order $m - 1 < \alpha \leq m$ ($m \in \mathbb{N}$) with respect to t is defined in [1] as

$$\left(\frac{{}_c\partial^\alpha}{\partial t^\alpha} u\right)(t, x) = \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - \tau)^{m-\alpha-1} u_\tau^{(m)}(\tau, x) d\tau.$$

In particular, for $1 < \alpha \leq 2$,

$$\left(\frac{{}_c\partial^\alpha}{\partial t^\alpha} u\right)(t, x) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t (t - \tau)^{1-\alpha} u_\tau^{(2)}(\tau, x) d\tau.$$

In addition, we define the fractional partial integral operator I_t^α of a function $u(t, x)$ for $\alpha \geq 0$ as

$$I_t^\alpha u(t, x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau, x) d\tau.$$

In particular,

$$I_t^0 u(t, x) = u(t, x).$$

1.2. Motivation

Fractional partial integrals and derivatives are fundamentally non-local operators, meaning they incorporate information over an entire domain rather than just a neighborhood of a point (like classical derivatives). This nonlocality is crucial for modeling systems where long-range interactions, memory effects, or spatially distributed dependencies play a key role.

In this paper, we will study the following equation with initial conditions and variable coefficients for $1 < \alpha \leq 2$ and $T > 0$:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) + f(t) \triangle u(t,x) = g(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x), \end{cases} \tag{1}$$

which, to the best of our knowledge, has not been previously studied.

This equation is essential in applications due to the following factors:

(a) The equation involves a Caputo fractional derivative of order $1 < \alpha \leq 2$, which generalizes the classical second-order time derivative. This allows it to model systems that exhibit memory and hereditary effects, which classical diffusion or wave equations cannot capture.

(b) This is a fractional diffusion–wave equation, interpolating between diffusion ($\alpha = 1$) and wave ($\alpha = 2$) dynamics. It is applicable in anomalous diffusion, viscoelastic materials, signal processing, and finance.

(c) The function $f(t)$ introduces time-dependent diffusivity, making the model more realistic for non-homogeneous media where the material properties evolve with time.

(d) Solving such equations requires advanced tools, like the inverse operator method, especially when $f(t)$ or $g(t,x)$ are not constant. These methods help to establish the existence, uniqueness, and even series representations of solutions.

Furthermore, the time–fractional wave equation has emerged as a central topic in the study of anomalous diffusion and wave propagation phenomena, particularly where memory and hereditary properties simulate. Unlike classical wave equations, time–fractional models incorporate nonlocality in time, typically via the Caputo or Riemann–Liouville fractional derivatives, enabling them to describe complex physical processes more accurately. This model generalizes the classical wave and heat equations, allowing for the description of anomalous diffusion and memory-dependent processes in complex systems. The presence of variable coefficients makes the equation suitable for real-world applications in heterogeneous media, while also introducing significant analytical challenges. The inclusion of first- and second-order initial conditions further enhances its applicability to physical problems involving memory and inertia. The inverse operator method applied in this paper offers a robust approach to establishing the existence and uniqueness of a series solution, making this study both mathematically rich and practically relevant.

1.3. Literature Review

One of the foundational contributions is made by Mainardi [3,4], who investigated the fundamental solutions of the time–fractional diffusion–wave equation, revealing its intermediate nature between parabolic and hyperbolic dynamics. Gorenflo et al. [5] further analyzed these fundamental solutions using Fox H-functions, providing insights into their analytical structures. Podlubny [6] provided a comprehensive mathematical framework for fractional differential equations, laying the groundwork for further theoretical advancements.

Theoretical studies have focused on the existence, uniqueness, and regularity of solutions. Eidelman and Kochubei [7] analyzed the Cauchy problem for fractional diffusion–wave equations, providing integral representations and exploring the properties of fundamental solutions. Similarly, Luchko [8] established maximum principles and uniqueness theorems for time–fractional PDEs, extending classical results to the fractional setting.

Spectral methods and eigenfunction expansions have been employed for analytical solutions, especially in bounded domains. Li and Liu [9] investigated eigenfunction expansions for multi-term time–fractional wave equations, contributing to the spectral theory of fractional models. Moreover, Kilbas et al. [1] provided detailed theoretical insights into the mathematical properties of fractional operators and their applications to linear and nonlinear PDEs.

In 2022, Abuomar et al. [10] studied the following fractional wave equation with certain conditions for $0.5 < \alpha < 1$:

$$\frac{{}_c\partial^{2\alpha}}{\partial t^{2\alpha}} u(t,x,y) = u_{xx}(t,x,y) + u_{yy}(t,x,y) + g(t,x,y), \;\; 0 < x, y < \infty, \; t > 0,$$

based on the Laplace transform and fractional series method.

Very recently, Li and Liao [11] studied the generalized time–fractional diffusion–wave equation for $1 < \alpha \leq 2$ using the inverse operator method for all constants λ_j and $1 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < \alpha \leq 2$:

$$\begin{cases} \frac{{}_c\partial^{\alpha}}{\partial t^{\alpha}} u(t,x) + \sum\limits_{j=1}^{m} \lambda_j \frac{{}_c\partial^{\alpha_j}}{\partial t^{\alpha_j}} u(t,x) = \triangle u(t,x) + g(t,x), \\ u(0,x) = \theta(x), \quad u_t(0,x) = \beta(x), \;\; (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \;\; m \in \mathbb{N}, \end{cases}$$

based on the multivariate Mittag–Leffler function and a newly constructed space.

Huang and Yamamoto [12] established existence, uniqueness, and regularity estimates for fractional diffusion–wave equations (fractional order in $(1,2]$) with time-dependent coefficients, via Fredholm and Galerkin methods. Li et al. [13] derived a unique series solution to a multiple time–fractional convection diffusion equation with a non-homogenous source term based on an inverse operator, a newly constructed space, and the multivariate Mittag–Leffler function. Ge and Zhang [14] investigated the exponential/polynomial stability of wave equations with boundary fractional damping and spatially variable coefficients.

There are both analytical approaches [15]—such as convolution and fractional Green's functions, various types of integral transforms, separation of variables, the Adomian decomposition method, and the homotopy analysis method—and numerical methods [6], including finite difference methods, finite element methods, spectral methods, and meshless methods, for solving fractional partial differential equations.

1.4. The Klein–Gordon Equation and Research Methods

The motivation for using the Mittag–Leffler function and the inverse operator method in this paper stems from the fact that Equation (1) is not easily handled by any currently existing theoretical methods—including Laplace and Fourier transforms—due to the presence of the variable coefficient $f(t)$, which is independent of any spatial variable. In addition, this approach is also useful for finding analytical solutions to fractional partial differential equations and partial integro-differential equations under various conditions [11]. As a demonstration, we first apply this technique to the following Klein–Gordon equation, as a preparatory step toward solving Equation (1).

Let S be the subspace of $C^\infty(\mathbb{R}^n)$, given by

$$S = \Big\{ \psi \in C^\infty(\mathbb{R}^n) : \text{ for any } k \in \mathbb{N} \cup \{0\} \;\; \exists \text{ a constant } M_\psi > 0 \text{ and a positive function } \\ \theta(x) \in C(\mathbb{R}^n) \text{ such that } \left| \triangle^k \psi(x) \right| \leq \theta(x) M_\psi^k \Big\}.$$

Theorem 1. *Let $T > 0$. We assume that $u_t(t,x)$ is continuous over $[0,T]$ in t and $u_{tt}(t,x) \in L^1[0,T]$. Then, the following time–fractional Klein–Gordon equation for $1 < \alpha \leq 2$ and $u(t,x)$ is a smooth function over $\mathbb{R}^n$ with respect to x:*

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) - \triangle u(t,x) + m^\alpha u(t,x) = 0, \\ u(0,x) = \psi_1(x), \quad u_t(0,x) = \psi_2(x), \quad (t,x) \in [0,T] \times \mathbb{R}^n, \end{cases} \tag{2}$$

where $m \geq 0$ is a constant (mass parameter) that admits a unique solution,

$$\begin{aligned} u(t,x) = &\sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \sum_{k_1 + k_2 = k} \binom{k}{k_1, k_2} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_1(x) \\ &+ \sum_{k=0}^{\infty} \frac{t^{\alpha k + 1}}{\Gamma(\alpha k + 2)} \sum_{k_1 + k_2 = k} \binom{k}{k_1, k_2} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_2(x), \end{aligned} \tag{3}$$

if ψ_1 and ψ_2 are in S. Furthermore, there is an integral solution:

$$u(t,x) = \int_{\mathbb{R}^n} \psi_1(y) G_1(t, x-y) dy + t \int_{\mathbb{R}^n} \psi_2(y) G_2(t, x-y) dy,$$

where the Green's functions are defined by

$$G_1(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_\alpha\Big(\big(-|\zeta|^2 - m^\alpha\big) t^\alpha\Big) e^{i\langle \zeta, x\rangle} d\zeta,$$

and

$$G_2(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_{\alpha,2}\Big(\big(-|\zeta|^2 - m^\alpha\big) t^\alpha\Big) e^{i\langle \zeta, x\rangle} d\zeta.$$

Proof. Clearly, for $1 < \alpha \leq 2$,

$$\frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2}{\partial s^2} u(s,x) ds = I_t^{2-\alpha} u_{tt}(t,x),$$

provided $u_t(t,x)$ is continuous over $[0,T]$ in t and $u_{tt}(t,x) \in L^1[0,T]$. Then,

$$I_t^\alpha \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) = I_t^\alpha (I_t^{2-\alpha} u_{tt})(t,x) = (I_t^2 u_{tt})(t,x) = u(t,x) - u(0,x) - u_t(0,x)t.$$

Applying the operator I_t^α to both sides of Equation (2) yields

$$u(t,x) - u(0,x) - u_t(0,x)t - I_t^\alpha \triangle u(t,x) + m^\alpha I_t^\alpha u(t,x) = 0,$$

which implies that

$$(1 - I_t^\alpha \triangle + m^\alpha I_t^\alpha) u(t,x) = \psi_1(x) + \psi_2(x)t,$$

by using the initial conditions.

To find the inverse operator of $1 - I_t^\alpha \triangle + m^\alpha I_t^\alpha$, we define an operator V as

$$\begin{aligned} V = \sum_{k=0}^{\infty} (I_t^\alpha \triangle - m^\alpha I_t^\alpha)^k &= \sum_{k=0}^{\infty} \sum_{k_1 + k_2 = k} \binom{k}{k_1, k_2} (I_t^\alpha \triangle)^{k_1} (-m^\alpha I_t^\alpha)^{k_2} \\ &= \sum_{k=0}^{\infty} \sum_{k_1 + k_2 = k} \binom{k}{k_1, k_2} (-m^\alpha)^{k_2} I_t^{\alpha k_1 + \alpha k_2} \triangle^{k_1}, \end{aligned}$$

which is well defined over the space S. Indeed, for any $\psi \in S$, we have

$$\left| \triangle^{k_1} \psi(x) \right| \leq \theta(x) M_\psi^{k_1},$$

for any non-negative integer k_1. Thus,

$$\begin{aligned} |V\psi| &\leq \theta(x) \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} |m|^{\alpha k_2} \frac{t^{\alpha k_1 + \alpha k_2}}{\Gamma(\alpha k_1 + \alpha k_2 + 1)} M_\psi^{k_1} \\ &= \theta(x) E_{(\alpha,\alpha),\,1}\left(M_\psi t^\alpha, |m|^\alpha t^\alpha\right) < +\infty \end{aligned}$$

for each $(t, x) \in [0, T] \times \mathbb{R}^n$. Furthermore,

$$V(1 - I_t^\alpha \triangle + m^\alpha I_t^\alpha) = (1 - I_t^\alpha \triangle + m^\alpha I_t^\alpha)V = 1 \text{ (identity)}.$$

It follows that

$$\begin{aligned} &V(1 - I_t^\alpha \triangle + m^\alpha I_t^\alpha) = V - V(I_t^\alpha \triangle - m^\alpha I_t^\alpha) \\ &= 1 + \sum_{k=1}^{\infty} (I_t^\alpha \triangle - m^\alpha I_t^\alpha)^k - \sum_{k=0}^{\infty} (I_t^\alpha \triangle - m^\alpha I_t^\alpha)^{k+1} = 1. \end{aligned}$$

Similarly,

$$(1 - I_t^\alpha \triangle + m^\alpha I_t^\alpha)V = 1$$

and the uniqueness of V is clear. Hence,

$$\begin{aligned} u(t, x) &= V(\psi_1(x) + \psi_2(x)t) \\ &= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2}}{\Gamma(\alpha k_1 + \alpha k_2 + 1)} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_1(x) \\ &\quad + \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2 + 1}}{\Gamma(\alpha k_1 + \alpha k_2 + 2)} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_2(x). \end{aligned}$$

Using the Fourier transform, we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2}}{\Gamma(\alpha k_1 + \alpha k_2 + 1)} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_1(x) \\ &= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2}}{\Gamma(\alpha k_1 + \alpha k_2 + 1)} (-m^\alpha)^{k_2} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-|\zeta|^2)^{k_1} \tilde{\psi}_1(\zeta) \mathrm{e}^{\mathrm{i}\langle \zeta, x \rangle} d\zeta \\ &= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2}}{\Gamma(\alpha k_1 + \alpha k_2 + 1)} (-m^\alpha)^{k_2} \\ &\cdot \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-|\zeta|^2)^{k_1} \left(\int_{\mathbb{R}^n} \psi_1(x) \mathrm{e}^{-\mathrm{i}\langle \zeta, x \rangle} dx \right) \mathrm{e}^{\mathrm{i}\langle \zeta, x \rangle} d\zeta. \end{aligned}$$

Since

$$\sum_{k_1+k_2=k} \binom{k}{k_1, k_2} (-|\zeta|^2)^{k_1} (-m^\alpha)^{k_2} = \left(-|\zeta|^2 - m^\alpha\right)^k,$$

we deduce that

$$
\begin{aligned}
V\psi_1(x) &= \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(-|\zeta|^2 - m^\alpha\right)^k \left(\int_{\mathbb{R}^n} \psi_1(x) \mathrm{e}^{-\mathrm{i}\langle \zeta, x\rangle} dx\right) \mathrm{e}^{\mathrm{i}\langle \zeta, x\rangle} d\zeta \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_\alpha\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right) \left(\int_{\mathbb{R}^n} \psi_1(x) \mathrm{e}^{-\mathrm{i}\langle \zeta, x\rangle} dx\right) \mathrm{e}^{\mathrm{i}\langle \zeta, x\rangle} d\zeta \\
&= \mathcal{F}^{-1}\left[E_\alpha\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right) \left(\int_{\mathbb{R}^n} \psi_1(x) \mathrm{e}^{-\mathrm{i}\langle \zeta, x\rangle} dx\right)\right] \\
&= \mathcal{F}^{-1}\left[E_\alpha\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right)\right] * \psi_1(x).
\end{aligned}
$$

Let the Green function $G_1(t, x)$ be defined as

$$
G_1(t, x) = \mathcal{F}^{-1}\left[E_\alpha\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right)\right] = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_\alpha\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right) \mathrm{e}^{\mathrm{i}\langle \zeta, x\rangle} d\zeta.
$$

Then,

$$
V\psi_1(x) = \int_{\mathbb{R}^n} \psi_1(y) G_1(t, x-y) dy.
$$

On the other hand,

$$
\begin{aligned}
V\psi_2(x)t &= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2 + 1}}{\Gamma(\alpha k_1 + \alpha k_2 + 2)} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_2(x) \\
&= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2 + 1}}{\Gamma(\alpha k_1 + \alpha k_2 + 2)} (-m^\alpha)^{k_2} \\
&\cdot \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-|\zeta|^2)^{k_1} \left(\int_{\mathbb{R}^n} \psi_2(x) \mathrm{e}^{-\mathrm{i}\langle \zeta, x\rangle} dx\right) \mathrm{e}^{\mathrm{i}\langle \zeta, x\rangle} d\zeta \\
&= t\mathcal{F}^{-1}\left[E_{\alpha,\,2}\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right)\right] * \psi_2(x),
\end{aligned}
$$

using

$$
\sum_{k=0}^{\infty} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \left(-|\zeta|^2 - m^\alpha\right)^k = t \sum_{k=0}^{\infty} \frac{\left((-|\zeta|^2 - m^\alpha) t^\alpha\right)^k}{\Gamma(\alpha k+2)} = t E_{\alpha,2}\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right).
$$

Let the Green function $G_2(t, x)$ be defined as

$$
G_2(t, x) = \mathcal{F}^{-1}\left[E_{\alpha.2}\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right)\right] = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_{\alpha,2}\left(\left(-|\zeta|^2 - m^\alpha\right)^k t^\alpha\right) \mathrm{e}^{\mathrm{i}\langle \zeta, x\rangle} d\zeta.
$$

Then,

$$
V\psi_2(x) = t \int_{\mathbb{R}^n} \psi_2(y) G_2(t, x-y) dy.
$$

The uniqueness of solutions follows immediately from the uniqueness of the inverse operator V. This completes the proof. □

1.5. The Importance of Klein–Gordon Equation

The time–fractional Klein–Gordon Equation (2) is a significant generalization of the classical Klein–Gordon equation, which serves as a cornerstone in relativistic quantum mechanics and field theory. The fractional formulation is important for a variety of reasons—spanning mathematical, physical, and applied contexts:

(a) The key difference lies in the Caputo fractional derivative of order $1 < \alpha \leq 2$ in time, as follows: $\frac{{}^c\partial^\alpha}{\partial t^\alpha} u(t, x)$. This operator introduces memory effects, meaning that the future evolution of $u(t, x)$ depends not just on its current state but also on its entire history. This is critical for viscoelastic materials, non-Markovian processes in quantum mechanics, and anomalous diffusion (where the mean square displacement grows nonlinearly in time).

(b) The classical Klein–Gordon equation ($\alpha = 2$) models relativistic wave propagation. Replacing the second-order time derivative with a fractional derivative allows the model to capture memory effects and non-local temporal behavior—features typical of complex or disordered media. In addition, the time–fractional Klein–Gordon equation offers new insights in mathematical physics, especially in the study of non-local and fractional dynamics, and it drives the development of new analytical and numerical methods [16].

(c) Fractional PDEs like (2) open new avenues in functional analysis (especially in fractional Sobolev spaces), existence, uniqueness, and regularity of solutions, and spectral theory, especially for the fractional Laplacian and damping/memory terms. Theorem 1 asserts the well-posedness (existence and uniqueness under smooth initial conditions) of the problem, which is critical for building further analysis.

(d) The time–fractional Klein–Gordon equation appears in quantum field models with temporal dispersion or dissipation, acoustics in lossy media, plasma physics, where wave propagation is not purely classical, and geophysics (e.g., seismic wave attenuation).

In addition, the mass parameter m is typically taken to be real and non-negative for physical reasons: The Klein–Gordon equation describes spin-0 particles (scalar or pseudoscalar bosons) in quantum field theory, and the parameter m corresponds to the rest mass of the particle, which is a measurable physical quantity. Mass is conventionally non-negative in classical and quantum physics.

1.6. Further Discussion and Examples

Remark 1. *(a) Clearly, the solution u given in Equation (3) satisfies the conditions that $u_{tt}(t, x) \in L^1[0, T]$ and $u_t(t, x) \in C[0, T]$ by noting that $1 < \alpha \leq 2$.*

(b) In addition, in order to prove the equivalence between Equation (1) and Equation (3), we used the identity for $m - 1 < \alpha \leq m \in \mathbb{N}$

$$_C D^\alpha I^\alpha w(t) = w(t)$$

if $w \in C(0, T]$. Indeed, we are able to provide the following distributional proof directly. Let

$$k_\alpha(t) = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha \geq 0.$$

In particular, $k_0(t) = \delta(t)$ in the distributional sense, and

$$(\delta(t), \phi(t)) = \phi(0),$$

where ϕ is a smooth function with compact support [17]. Using

$$\frac{d^m}{dt^m} k_\alpha(t) = k_{\alpha-m}(t)$$

implies that

$$\frac{d^m}{dt^m}(k_\alpha * w) = k_{\alpha-m} * w.$$

Thus,

$$_C D^\alpha I^\alpha w = k_{m-\alpha} * (k_{\alpha-m} * w) = (k_{m-\alpha} * k_{\alpha-m}) * w = \delta * w = w,$$

by using the identity

$$k_{m-\alpha} * k_{\alpha-m} = k_0(t) = \delta(t).$$

Since w is continuous, the identity

$$ {}_C D^\alpha I^\alpha w(t) = w(t) $$

holds point-wise on the interval $(0, T]$.

Remark 2. *Following the similar technique, we are able to find the unique series or integral solution for the time–fractional non-homogeneous Klein–Gordon equation:*

$$ \begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) - \triangle u(t,x) + m^\alpha u(t,x) = g(t,x), \\ u(0,x) = \psi_1(x), \quad u'_t(0,x) = \psi_2(x), \quad (t,x) \in [0,T] \times \mathbb{R}^n, \end{cases} $$

with a slight modification of the space S and certain conditions on g:

$$ S_M = \Big\{ g(t,x) \in C([0,T] \times \mathbb{R}^n) : \text{ for any } k \in \mathbb{N} \cup \{0\} \ \exists \text{ a constant } M_g > 0 \text{ and a positive function } \theta(t,x) \in C([0,T] \times \mathbb{R}^n) \text{ such that } \left| \triangle^k g(t,x) \right| \le \theta(t,x) M_g^k \Big\}. $$

Then,

$$ \begin{aligned} u(t,x) = & \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} (-m^\alpha)^{k_2} I_t^{\alpha k_1 + \alpha k_2 + \alpha} \triangle^{k_1} g(t,x) \\ & \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2}}{\Gamma(\alpha k_1 + \alpha k_2 + 1)} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_1(x) \\ & + \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{t^{\alpha k_1 + \alpha k_2 + 1}}{\Gamma(\alpha k_1 + \alpha k_2 + 2)} (-m^\alpha)^{k_2} \triangle^{k_1} \psi_2(x) \\ = & \int_0^t \int_{\mathbb{R}^n} G_0(t-s, x-y) g(s,y) dy ds + \int_{\mathbb{R}^n} \psi_1(y) G_1(t, x-y) dy \\ & + t \int_{\mathbb{R}^n} \psi_2(y) G_2(t, x-y) dy, \end{aligned} $$

where the Green's functions G_1 and G_2 are defined above, and

$$ G_0(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_\alpha \Big(\big(-|\zeta|^2 - m^\alpha \big) t^\alpha \Big) e^{i \langle \zeta, x \rangle} d\zeta, $$

which is the fundamental solution to equation:

$$ \begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) - \triangle u(t,x) + m^\alpha u(t,x) = 0, \\ u(0,x) = \delta(x), \quad u_t(0,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n. \end{cases} $$

Example 1. *The following time–fractional Klein–Gordon equation for $1 < \alpha \le 2$,*

$$ \begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) - \frac{\partial^2}{\partial x^2} u(t,x) + u(t,x) = 0, \\ u(0,x) = \sin x, \quad u_t(0,x) = 1, \quad (t,x) \in [0,T] \times \mathbb{R}, \end{cases} $$

has a unique solution:

$$ u(t,x) = \sin x E_\alpha(-2t^\alpha) + t E_{\alpha,2}(-t^\alpha). $$

Evidently, $\sin x$ and 1 are in the space S. From Theorem 1, we have

$$u(t,x) = \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} \frac{t^{\alpha k_1+\alpha k_2}}{\Gamma(\alpha k_1+\alpha k_2+1)} (-1)^{k_2} \frac{d^{2k_1}}{dx^{2k_1}} \sin x$$
$$+ \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} \frac{t^{\alpha k_1+\alpha k_2+1}}{\Gamma(\alpha k_1+\alpha k_2+2)} (-1)^{k_2} \frac{d^{2k_1}}{dx^{2k_1}} 1.$$

Applying the identity

$$\frac{d^{2k_1}}{dx^{2k_1}} \sin x = \sin(x+k_1\pi) = (-1)^{k_1} \sin x,$$

we arrive at

$$\begin{aligned} u(t,x) &= \sin x \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1))} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} (-1)^{k_1}(-1)^{k_2} + \sum_{k=0}^{\infty} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} (-1)^k \\ &= \sin x \sum_{k=0}^{\infty} \frac{t^{\alpha k}(-2)^k}{\Gamma(\alpha k+1)} + \sum_{k=0}^{\infty} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} (-1)^k = \sin x E_\alpha(-2t^\alpha) + tE_{\alpha,2}(-t^\alpha) \\ &= \sin x - \frac{2t^\alpha}{\Gamma(\alpha+1)} \sin x + \sin x \sum_{k=2}^{\infty} \frac{t^{\alpha k}(-2)^k}{\Gamma(\alpha k+1)} + t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{k=2}^{\infty} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} (-1)^k, \end{aligned}$$

which satisfies the initial conditions

$$u(0,x) = \sin x, \quad u_t(0,x) = 1,$$

by noting that $\alpha > 1$.

Clearly, this series approach is much simpler than using the integral solution which contains complicated computations.

For the special case where $\alpha = 2$, the solution to Equation (2) is

$$u(t,x) = \sin x \frac{\sin(\sqrt{2}t)}{\sqrt{2}t} + \frac{1-\cos t}{t},$$

by the identities

$$E_2(-t) = \frac{\sin\sqrt{t}}{\sqrt{t}}, \quad E_{2,2}(-t) = \frac{1-\cos\sqrt{t}}{t},$$

for $t > 0$.

In general, the Klein–Gordon equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) - \triangle u(t,x) + m^2 u(t,x) = 0, \\ u(0,x) = \psi_1(x), \quad u_t'(0,x) = \psi_2(x), \quad (t,x) \in [0,T] \times \mathbb{R}^n, \end{cases}$$

has the solution

$$\begin{aligned}
u(t,x) &= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} \frac{t^{2k}}{\Gamma(2k+1)} (-m^2)^{k_2} \triangle^{k_1} \psi_1(x) \\
&\quad + \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1,k_2} \frac{t^{2k+1}}{\Gamma(2k+2)} (-m^2)^{k_2} \triangle^{k_1} \psi_2(x) \\
&= E_2(t^2(\triangle - m^2))\psi_1(x) + tE_{2,2}(t^2(\triangle - m^2))\psi_2(x) \\
&= \int_0^1 \cos\left(ts\sqrt{\triangle - m^2}\right)\psi_1(x)ds + \int_0^t \frac{\sin((t-s)\sqrt{\triangle - m^2})}{\sqrt{\triangle - m^2}} \psi_2(x)ds,
\end{aligned}$$

using the identities [6]:

$$E_1(t^2A) = \int_0^1 \cos(ts\sqrt{A})ds, \;\; tE_{2,2}(t^2A) = \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} ds.$$

2. The Fractional Wave Equation with Variable Coefficients

In this section, we mainly work on Equation (1) based on the inverse operator method, the Mittag–Leffler function and a new space S_0, with two illustrative examples.

Theorem 2. *We assume that $u_t(t,x)$ is continuous over $[0,T]$ in t and $u_{tt}(t,x) \in L^1[0,T]$. Let $f \in C(\mathbb{R}^+)$ and g, ϕ and ψ be in the space S_0 defined by*

$$\begin{aligned}
S_0 = &\{g \in C([0,T]\times\mathbb{R}^n) : \textit{for any } k \in \mathbb{N}\cup\{0\} \;\exists \textit{ a constant } M_g > 0 \textit{ and a positive} \\
&\textit{function } \theta_0(t,x) \in C([0,T]\times\mathbb{R}^n) \textit{ such that } \left|\triangle^k g(t,x)\right| \le \theta_0(t,x) M_g^k\}.
\end{aligned}$$

Then, the solution $u(t,x)$ to Equation (1), being a smooth function over $\mathbb{R}^n$ with respect to x, admits a unique series solution:

$$\begin{aligned}
u(t,x) &= \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t))^k \; I_t^\alpha \; \triangle^k g(t,x) + \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t))^k \; \triangle^k \phi(x) \\
&\quad + \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t))^k \; t \; \triangle^k \psi(x).
\end{aligned}$$

In particular, if $f(t) = c_0$ (constant), then

$$\begin{aligned}
u(t,x) &= \sum_{k=0}^{\infty} (-1)^k c_0^k I_t^{\alpha k+\alpha} \; \triangle^k g(t,x) + \sum_{k=0}^{\infty} (-1)^k \; c_0^k \; \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \; \triangle^k \phi(x) \\
&\quad + \sum_{k=0}^{\infty} (-1)^k \; c_0^k \; \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \; \triangle^k \psi(x) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} I_t^\alpha \left[\sum_{k=0}^{\infty} \left(c_0|\zeta|^2 I_t^\alpha\right)^k \tilde{g}(t,\zeta) \right] e^{i\langle\zeta,x\rangle} d\zeta \\
&\quad + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_\alpha\left(c_0|\zeta|^2 t^\alpha\right) \tilde{\phi}(\zeta) e^{i\langle\zeta,x\rangle} d\zeta + \frac{t}{(2\pi)^n} \int_{\mathbb{R}^n} E_{\alpha,2}\left(c_0|\zeta|^2 t^\alpha\right) \tilde{\psi}(\zeta) e^{i\langle\zeta,x\rangle} d\zeta,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{g}(t,\zeta) &= \int_{\mathbb{R}^n} g(t,x) e^{-i\langle\zeta,x\rangle} dx, \\
\tilde{\phi}(\zeta) &= \int_{\mathbb{R}^n} \phi(x) e^{-i\langle\zeta,x\rangle} dx, \;\; \tilde{\psi}(\zeta) = \int_{\mathbb{R}^n} \psi(x) e^{-i\langle\zeta,x\rangle} dx.
\end{aligned}$$

Proof. By applying the operator I_t^α to both sides of Equation (1), we have

$$u(t,x) - \phi(x) - \psi(x)t + I_t^\alpha f(t) \triangle u(t,x) = I_t^\alpha g(t,x),$$

which implies that

$$(1 + I_t^\alpha f(t) \triangle) u(t,x) = I_t^\alpha g(t,x) + \phi(x) + \psi(x)t.$$

To find a unique inverse operator of $1 + I_t^\alpha f(t) \triangle$ over S_0, we define

$$V_0 = \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t) \triangle)^k = \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t))^k \triangle^k$$

which is well defined over S_0. Indeed, for $g \in S_0$, we get

$$\left| \triangle^k g(t,x) \right| \leq \theta_0(t,x) M_g^k,$$

and

$$\left| (I_t^\alpha f(t))^k \theta_0(t,x) \right| \leq \max_{\tau \in [0,T]} \theta_0(\tau,x) \max_{\tau \in [0,T]} |f(\tau)|^k \frac{T^{\alpha k}}{\Gamma(\alpha k+1)},$$

using

$$I_t^{\alpha k} 1 = \frac{t^{\alpha k}}{\Gamma(\alpha k+1)}.$$

Therefore,

$$\begin{aligned} |V_0 g| &\leq \sum_{k=0}^{\infty} \left| (I_t^\alpha f(t))^k \right| |\triangle^k g(t,x)| \leq \max_{\tau \in [0,T]} \theta_0(\tau,x) \sum_{k=0}^{\infty} \max_{\tau \in [0,T]} |f(\tau)|^k \frac{T^{\alpha k}}{\Gamma(\alpha k+1)} M_g^k \\ &= \max_{\tau \in [0,T]} \theta_0(\tau,x) E_\alpha \left(\max_{\tau \in [0,T]} |f(\tau)| T^\alpha M_g \right) < +\infty, \end{aligned}$$

by noting that $f \in C(\mathbb{R}^+)$. Moreover, V_0 is a unique inverse operator of $1 + I_t^\alpha f(t) \triangle$ since

$$V_0 (1 + I_t^\alpha f(t) \triangle) = (1 + I_t^\alpha f(t) \triangle) V_0 = 1.$$

In fact,

$$\begin{aligned} V_0 (1 + I_t^\alpha f(t) \triangle) &= V_0 + \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t) \triangle)^{k+1} \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k (I_t^\alpha f(t) \triangle)^k + \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t) \triangle)^{k+1} = 1. \end{aligned}$$

Similarly,

$$(1 + I_t^\alpha f(t) \triangle) V_0 = 1,$$

and the uniqueness of V_0 follows easily. Hence,

$$\begin{aligned} u(t,x) = V_0 (I_t^\alpha g(t,x) + \phi(x) + \psi(x)t) &= \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t))^k \, I_t^\alpha \triangle^k g(t,x) \\ &+ \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t))^k \triangle^k \phi(x) + \sum_{k=0}^{\infty} (-1)^k (I_t^\alpha f(t))^k \, t \, \triangle^k \psi(x). \end{aligned}$$

If $f(t)=c_0$, then

$$u(t,x)=\sum_{k=0}^{\infty}(-1)^k c_0^k I_t^{\alpha k+\alpha} \triangle^k g(t,x)+\sum_{k=0}^{\infty}(-1)^k\ c_0^k\ \frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\ \triangle^k\phi(x)$$
$$+\sum_{k=0}^{\infty}(-1)^k\ c_0^k\ \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}\ \triangle^k\psi(x)=I_1+I_2+I_3.$$

Firstly, we consider I_1, which is

$$I_1=\sum_{k=0}^{\infty}(-1)^k c_0^k I_t^{\alpha k+\alpha}\ \triangle^k g(t,x)=\sum_{k=0}^{\infty}(-1)^k c_0^k I_t^{\alpha k+\alpha}$$
$$\cdot\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(-|\zeta|^2)^k\tilde{g}(t,\zeta)\mathrm{e}^{\mathrm{i}\langle\zeta,x\rangle}d\zeta$$
$$=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}I_t^{\alpha}\left[\sum_{k=0}^{\infty}\left(c_0|\zeta|^2 I_t^{\alpha}\right)^k\tilde{g}(t,\zeta)\right]\mathrm{e}^{\mathrm{i}\langle\zeta,x\rangle}d\zeta,$$

by using

$$I_t^{\alpha k+\alpha}=I_t^{\alpha}\cdot I_t^{\alpha k}.$$

As for

$$I_2=\sum_{k=0}^{\infty}(-1)^k\ c_0^k\ \frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\ \triangle^k\phi(x)$$
$$=\sum_{k=0}^{\infty}(-1)^k\ c_0^k\ \frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(-|\zeta|^2\right)^k\tilde{\phi}(\zeta)\mathrm{e}^{\mathrm{i}\langle\zeta,x\rangle}d\zeta$$
$$=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}E_{\alpha}\left(c_0|\zeta|^2t^{\alpha}\right)\tilde{\phi}(\zeta)\mathrm{e}^{\mathrm{i}\langle\zeta,x\rangle}d\zeta.$$

Finally,

$$I_3=\sum_{k=0}^{\infty}(-1)^k\ c_0^k\ \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}\ \triangle^k\psi(x)$$
$$=\sum_{k=0}^{\infty}(-1)^k\ c_0^k\ \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(-|\zeta|^2\right)^k\tilde{\psi}(\zeta)\mathrm{e}^{\mathrm{i}\langle\zeta,x\rangle}d\zeta$$
$$=\frac{t}{(2\pi)^n}\int_{\mathbb{R}^n}E_{\alpha,2}\left(c_0|\zeta|^2t^{\alpha}\right)\tilde{\psi}(\zeta)\mathrm{e}^{\mathrm{i}\langle\zeta,x\rangle}d\zeta.$$

From the solution

$$u(t,x)=\sum_{k=0}^{\infty}(-1)^k(I_t^{\alpha}f(t))^k\ I_t^{\alpha}\ \triangle^k g(t,x)+\sum_{k=0}^{\infty}(-1)^k(I_t^{\alpha}f(t))^k\ \triangle^k\phi(x)$$
$$+\sum_{k=0}^{\infty}(-1)^k(I_t^{\alpha}f(t))^k\ t\ \triangle^k\psi(x),$$

we clearly see that

$$u(0,x)=\phi(x),\ \ u_t(0,x)=\psi(x),$$

since $\alpha>1$. This completes the proof. $\square$

Example 2. *The following is a fractional wave equation with a variable coefficient and initial conditions:*

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) + t^{1/2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u(t,x) = t \sin(2x_1)\cos(3x_2) \quad (t,x) \in [0,T] \times \mathbb{R}^2, \\ u(0,x) = J_0(ar), \quad u'_t(0,x) = x_1^2 - x_2^2, \quad r = (x_1^2 + x_2^2)^{1/2}, \end{cases} \tag{4}$$

where a is a constant and

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2} \right)^{2m}$$

is the Bessel function of the first kind, which admits a unique solution:

$$u(t,x) = \sin(2x_1)\cos(3x_2) \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \sum_{k=0}^{\infty} \frac{\pi^{k/2}}{2^k} \frac{t^{(\alpha+1/2)k}}{\Gamma^k(\alpha+3/2)} 13^k$$
$$+ J_0(ar) \sum_{k=0}^{\infty} \frac{\pi^{k/2}}{2^k} \frac{t^{(\alpha+1/2)k}}{\Gamma^k(\alpha+3/2)} a^{2k} + t(x_1^2 - x_2^2).$$

In $\mathbb{R}^2$, the Laplace operator in polar coordinates (r,θ) is

$$\triangle f = \frac{\partial^2}{\partial r^2} f + \frac{1}{r} \frac{\partial}{\partial r} f + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} f.$$

For a radially symmetric function $f = f(r)$, we obtain

$$\triangle f = f''(r) + \frac{1}{r} f'(r).$$

We now consider the eigenfunction equation for the Laplacian:

$$\triangle f + a^2 f = 0.$$

For a redial solution $f(r)$, this becomes

$$f''(r) + \frac{1}{r} f'(r) + a^2 f(r) = 0,$$

which is Bessel's equation for order $\nu = 0$, and its solution is $J_0(ar)$. This implies that

$$\triangle J_0(ar) = -a^2 J_0(ar).$$

Thus, applying the Laplacian k times gives

$$\triangle^k J_0(ar) = (-1)^k a^{2k} J_0(ar).$$

This identity also deduces that $J_0(ar) \in S_0$. On the other hand,

$$\triangle \sin(2x_1)\cos(3x_2) = -(2^2 + 3^2) \sin(2x_1)\cos(3x_2),$$

which claims that

$$\triangle^k \sin(2x_1)\cos(3x_2) = (-1)^k (2^2 + 3^2)^k \sin(2x_1)\cos(3x_2).$$

From Theorem 2, we get

$$u(t,x) = \sum_{k=0}^{\infty}(-1)^k \left(I_t^\alpha t^{1/2}\right)^k I_t^\alpha t \triangle^k \sin(2x_1)\cos(3x_2) + \sum_{k=0}^{\infty}(-1)^k \left(I_t^\alpha t^{1/2}\right)^k \triangle^k J_0(ar)$$
$$+ \sum_{k=0}^{\infty}(-1)^k \left(I_t^\alpha t^{1/2}\right)^k t \triangle^k (x_1^2 - x_2^2).$$

Using

$$I_t^\alpha t^{1/2} = \frac{\sqrt{\pi}}{2}\frac{t^{\alpha+1/2}}{\Gamma(\alpha+3/2)}, \quad I_t^\alpha t = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

we have

$$u(t,x) = \sin(2x_1)\cos(3x_2)\sum_{k=0}^{\infty}\frac{\pi^{k/2}}{2^k}\frac{t^{(\alpha+1/2)k}}{\Gamma^k(\alpha+3/2)}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}(2^2+3^2)^k$$
$$+ J_0(ar)\sum_{k=0}^{\infty}\frac{\pi^{k/2}}{2^k}\frac{t^{(\alpha+1/2)k}}{\Gamma^k(\alpha+3/2)}a^{2k} + t(x_1^2 - x_2^2),$$

by noting that $t\sin(2x_1)\cos(3x_2)$ and $x_1^2 - x_2^2$ are in the space S_0 and

$$\triangle^k(x_1^2 - x_2^2) = 0,$$

for all $k \geq 1$.

Remark 3. *We should note that the series*

$$\sum_{k=0}^{\infty}\frac{\pi^{k/2}}{2^k}\frac{t^{(\alpha+1/2)k}}{\Gamma^k(\alpha+3/2)}13^k, \quad t \in [0,T],$$

converges. Indeed,

$$\sum_{k=0}^{\infty}\frac{\pi^{k/2}}{2^k}\frac{t^{(\alpha+1/2)k}}{\Gamma^k(\alpha+3/2)}13^k = \sum_{k=0}^{\infty}\left(I_t^\alpha t^{1/2}\right)^k 13^k,$$

and

$$0 \leq \left(I_t^\alpha t^{1/2}\right)^k 13^k \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{1/2}d\tau \cdots \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{1/2}d\tau 13^k$$
$$\leq t^{k/2} I_t^{\alpha k} 13^k = t^{k/2}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}13^k.$$

Hence,

$$0 \leq \sum_{k=0}^{\infty}\frac{\pi^{k/2}}{2^k}\frac{t^{(\alpha+1/2)k}}{\Gamma^k(\alpha+3/2)}13^k \leq \sum_{k=0}^{\infty}\frac{(13t^{\alpha+1/2})^k}{\Gamma(\alpha k+1)} = E_\alpha\left(13t^{\alpha+1/2}\right)$$
$$\leq E_\alpha\left(13T^{\alpha+1/2}\right) < +\infty.$$

Remark 4. *If $\alpha = 2$ and $f(t) = -1$, then Equation (1) becomes a non-homogeneous wave equation:*

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x) = \triangle u(t,x) + g(t,x), & (t,x) \in [0,T]\times\mathbb{R}^n, \\ u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x), \end{cases} \tag{5}$$

with the solution

$$
\begin{aligned}
u(t,x) =& \sum_{k=0}^{\infty}(-1)^k (I_t^{\alpha} f(t))^k \, I_t^{\alpha} \triangle^k g(t,x) + \sum_{k=0}^{\infty}(-1)^k (I_t^{\alpha} f(t))^k \triangle^k \phi(x) \\
&+ \sum_{k=0}^{\infty}(-1)^k (I_t^{\alpha} f(t))^k \, t \triangle^k \psi(x) \\
=& \sum_{k=0}^{\infty} I_t^{2k+2} \triangle^k g(t,x) + \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \triangle^k \phi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \triangle^k \psi(x).
\end{aligned}
$$

Li and Liao [11] showed that if $g(t,x) = 0$ *and* $n = 1$, *then*

$$
\begin{aligned}
u(t,x) &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \frac{d^{2k}}{dx^{2k}} \phi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \frac{d^{2k}}{dx^{2k}} \psi(x) \\
&= \frac{\phi(x+t) + \phi(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(\zeta) d\zeta.
\end{aligned}
$$

This is d'Alembert's formula for a one-dimensional homogeneous wave equation.

Furthermore, if $g(t,x) = 0$ *and* $n = 3$, *then*

$$
\begin{aligned}
u(t,x) &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \triangle^k \phi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \triangle^k \psi(x) \\
&= \frac{\partial}{\partial t} \frac{1}{SA(B_3(0,1))} \left(t \int_{\partial B_3(0,1)} \phi(x + t\theta) ds(\theta) \right) + \frac{t}{SA(B_3(0,1))} \int_{\partial B_3(0,1)} \psi(x + t\theta) ds(\theta),
\end{aligned}
$$

which is the well-known Kirchoff formula [18], and where $SA(B_3(0,1))$ *denotes the surface area of the ball* $B_3(0,1)$ *and* $\partial B_3(0,1)$ *is the boundary of* $B_3(0,1)$.

We can use Kirchhoff's formula for the solution of the wave equation in three dimensions to derive the solution of the wave equation in two dimensions. This technique is known as the method of descent. A similar result also holds for $n = 2$.

Moreover, if $n > 3$ *and* n *is odd, then the series solution given is*

$$
u(t,x) = \frac{1}{c_n} \left(\frac{\partial}{\partial t} \right) \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(t^{n-2} \mathcal{A}_t \phi(x) \right) + \frac{1}{c_n} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(t^{n-2} \mathcal{A}_t \psi(x) \right),
$$

where

$$
c_n = 1 \cdot 3 \ldots (n-2),
$$

and the average value of ϕ *over* $\partial B_n(x,t)$ *is defined as*

$$
\mathcal{A}_t \phi(x) = \frac{1}{SA(B_n(x,t))} \int_{\partial B_n(x,t)} \phi(y) ds(y) = \frac{1}{SA(B_n(0,1))} \int_{\partial B_n(0,1)} \phi(x + t\theta) ds(\theta).
$$

A similar conclusion follows if $n > 3$ *and* n *is even.*

Example 3. *The following wave equation in* $\mathbb{R}^n$ *with the initial conditions,*

$$
\begin{cases}
\frac{\partial^2}{\partial t^2} u(t,x) = \triangle u(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\
u(0,x) = x_1 x_2, \quad u_t(0,x) = \sinh(x_1) \cdots \sinh(x_n), & n \in \mathbb{N},
\end{cases}
$$

has a unique solution:

$$
u(t,x) = x_1 x_2 + \sinh(x_1) \cdots \sinh(x_n) \frac{\sinh(t\sqrt{n})}{\sqrt{n}}.
$$

Clearly,

$$\triangle^k x_1 x_2 = 0,$$

for all $k \geq 1$ and

$$\triangle \sinh(x_1) \cdots \sinh(x_n) = n \sinh(x_1) \cdots \sinh(x_n),$$

which implies that

$$\triangle^k \sinh(x_1) \cdots \sinh(x_n) = n^k \sinh(x_1) \cdots \sinh(x_n).$$

From Theorem 2, we get

$$\begin{aligned} u(t,x) &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \triangle^k \phi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \triangle^k \psi(x) \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \triangle^k x_1 x_2 + \sinh(x_1) \cdots \sinh(x_n) \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} n^k \\ &= x_1 x_2 + \sinh(x_1) \cdots \sinh(x_n) t \sum_{k=0}^{\infty} \frac{(nt^2)^k}{(2k+1)!}. \end{aligned}$$

Using the identities

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}, \quad \sinh t = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!},$$

we get

$$\sinh(\sqrt{nt^2}) = \sinh(t\sqrt{n}) = t\sqrt{n} \sum_{k=0}^{\infty} \frac{(nt^2)^k}{(2k+1)!},$$

which claims that

$$\sum_{k=0}^{\infty} \frac{(nt^2)^k}{(2k+1)!} = \frac{\sinh(t\sqrt{n})}{t\sqrt{n}}.$$

Therefore,

$$u(t,x) = x_1 x_2 + \sinh(x_1) \cdots \sinh(x_n) \frac{\sinh(t\sqrt{n})}{\sqrt{n}},$$

which is a simple function.

3. Conclusions

By applying an inverse operator method, we studied the time–fractional wave equation (1) and the time–fractional Klein–Gordon equation (2), utilizing a multivariate Mittag–Leffler function and newly constructed function spaces. Moreover, we derived both series and integral solutions. In particular, our series-based technique provides a simpler process for finding solutions. This approach is also applicable to a wide range of fractional partial differential equations, such as the fractional Euler–Bernoulli beam equation:

$$\begin{cases} \frac{{}_c\partial^2}{\partial t^2} u(t,x) + \kappa \frac{{}_c\partial^{3/2}}{\partial t^{3/2}} u(t,x) + EI \int_0^t (t-\tau)^{-\nu} \frac{\partial^4 u(t,x)}{\partial x^4} d\tau = q(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x), \end{cases}$$

where $\nu \in (0,1)$.

This equation has good applications in viscoelastic beam vibrations with key features that combine fractional damping and stiffness.

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References

1. Kilbas, A.-A.; Srivastava, H.-M.; Trujillo, J.-J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
2. Hadid, S.-B.; Luchko, Y.-F. An operational method for solving fractional differential equations of an arbitrary real order. *Panamer. Math. J.* **1996**, *6*, 57–73.
3. Mainardi, F. The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett.* **1996**, *9*, 23–28. [CrossRef]
4. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity*; Imperial College Press: London, UK, 2010.
5. Gorenflo, R.; Loutchko, J.; Luchko, Y. Computation of the Mittag-Leffler function and fractional differential equations. *Fract. Calc. Appl. Anal.* **2002**, *5*, 491–518.
6. Podlubny, I. *Fractional Differential Equations*; Academie Press: New York, NY, USA, 1999.
7. Eidelman, S.D.; Kochubei, A.N. Cauchy problem for fractional diffusion equations. *J. Differ. Equ.* **2004**, *199*, 211–255. [CrossRef]
8. Luchko, Y. Maximum principle for the generalized time-fractional diffusion equation. *J. Math. Anal. Appl.* **2009**, *351*, 218–223. [CrossRef]
9. Li, C.; Liu, F. Spectral methods for multi-term time-fractional diffusion-wave equations. *Appl. Math. Model.* **2016**, *40*, 8220–8237.
10. Abuomar, M.M.A.; Syam, M.I.; Azmi, A. A study on fractional diffusion—Wave equation with a reaction. *Symmetry* **2022**, *14*, 1537. [CrossRef]
11. Li, C.; Liao, W. Applications of inverse operators to a fractional partial integro-differential equation and several well-known differential equations. *Fractal Fract.* **2025**, *9*, 200. [CrossRef]
12. Huang, J.; Yamamoto, M. Well-posedness of initial-boundary value problem for time-fractional diffusion/wave equation with time-dependent coefficients. *J. Evol. Equ.* **2025**, *25*, 68. [CrossRef]
13. Li, C.; Saadati, R.; Aderyani, S.R.; Luo, M.J. On the generalized fractional convection-diffusion equation with an initial condition in $\mathbb{R}^n$. *Fractal Fract.* **2025**, *9*, 347. [CrossRef]
14. Ge, S.; Zhang, W. Stability of wave equation with variable coefficients by boundary fractional dissipation law. *Results Math.* **2024**, *79*, 64. [CrossRef]
15. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Basel, Switzerland, 1993.
16. Tamsir, M.; Srivastava, V.K. Analytical study of time-fractional order Klein–Gordon equation. *Alex. Eng. J.* **2016**, *55*, 561–567. [CrossRef]
17. Li, C. Several results of fractional derivatives in $\mathcal{D}'(R_+)$. *Fract. Calc. Appl. Anal.* **2015**, *18*, 192–207. [CrossRef]
18. Strauss, W.A. *Partial Differential Equations: An Introduction*; Wiley: Hoboken, NJ, USA, 2007.