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# A uniqueness criterion for McKean–Vlasov fractional stochastic differential equations in $L_p$

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ABSTRACT

In this article, we establish a uniqueness criterion for McKean–Vlasov nonlinear stochastic differential equations in $L_p$, driven by Brownian motion. This is achieved using a combination of Nagumo-type and Osgood conditions. The main contribution of this paper is to unify various existing uniqueness theorems. Specifically, we extend the classical Lipschitz uniqueness theorem and incorporate recent results for ordinary differential equations in the case where $p = 2$, $\alpha = 1$, and distribution terms are omitted. Moreover, our methodology is applicable to nonlinear fractional stochastic differential equations (FSDEs). To demonstrate the applicability of our main result, we present numerical examples that support the theoretical findings.

## 1. Introduction

In recent years, the study of stochastic differential equations has become increasingly significant across a variety of fields. The challenge of ensuring uniqueness of solutions, especially in the presence of non-Lipschitz conditions, has attracted considerable attention from researchers. Uniqueness is a fundamental property in both theoretical and applied research, supporting the *reliability*, *accuracy*, and *practical applicability* of models across several scientific and engineering domains. The uniqueness of SDE solutions, for example, facilitates the precise simulation of risk factors, asset prices, and market dynamics in finance and insurance industries. Such accuracy is crucial for risk assessment, investment strategy development, and derivative pricing (see, for example, [1–3], and references therein).

In control theory and optimization, the existence of a unique solution supports the design of strategies and systems that reliably achieve desired goals. SDEs are used in biology and medicine to simulate inherently unpredictable processes such as population dynamics and disease transmission. Uniqueness ensures that the models accurately represent biological processes, enabling the design of effective intervention and treatment strategies.

Building on this significance, the main objective of this article is to study pathwise uniqueness for McKean–Vlasov fractional stochastic differential equations (FSDEs), an important subclass driven by Brownian motion in $L_p$ (the $p$th moment). The equation under consideration is given by:

$$ {}^{C}D_0^{\alpha} X(t) = f(t, X(t), \mathscr{L}_{X(t)}) + g(t, X(t), \mathscr{L}_{X(t)}) \frac{dB_t}{dt}, \quad X(0) = X_0, \quad t \in (0, T], \tag{1.1} $$

where the initial conditions include the derivatives $X^{(k)}(0) = X_0^{(k)}$ for $k = 1, 2, \ldots, \lfloor \alpha \rfloor$, in the case $\alpha \geq 1$. The operator ${}^{C}D_0^{\alpha}$ denotes the Caputo derivative of order $\alpha > 0$, taken at the origin, and will be formally defined later. Here, $X \in \mathbb{R}^n$ represents the unknown

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stochastic process, and $\mathscr{L}_{X(t)}$ indicates the law (i.e., the probability distribution) of $X(t)$. The coefficients $f : I \times \mathbb{R}^n \times \mathcal{M}_p(\mathbb{R}^n) \to \mathbb{R}^n$, and $g : I \times \mathbb{R}^n \times \mathcal{M}_p(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$ are assumed to be Borel measurable, where $I := [0, T]$, $\mathcal{M}_p(\mathbb{R}^n)$ is the set of probability measures on the Borel $\sigma$-algebra $\mathscr{B}(\mathbb{R}^n)$ with finite moment of order $p$. The process $B(t, \cdot)$ is an $m$-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

For such equations, the marginal distribution $\mathscr{L}_{X(t)}$ of the state variable $X(t)$ influences both the drift and diffusion coefficients. This feature is particularly useful for modeling real-world phenomena. These SDEs are especially effective for modeling complex systems in which the dynamics of individual components depend on the overall system state. For example, McKean–Vlasov SDEs are employed to describe the behavior of agents whose actions depend on the distribution of states across the entire population, such as in market and economic models.

Applications include models for pricing derivatives – where asset volatility depends on return distributions – and systemic risk, where an institution's stability is influenced by the overall financial environment. In neuroscience, they help describe the collective dynamics of neurons, and in epidemiology, they model disease transmission based on the population's health state. For further examples and discussions, see [4–7] and references therein.

Research on McKean–Vlasov SDEs has addressed various properties, including asymptotic behavior [8], exponential ergodicity [9], strong well-posedness [10–12], stability [13–15], and existence and uniqueness of solutions [16]. These foundational results have greatly advanced our understanding of the complex behaviors in such systems. However, works focusing on fractional-order McKean–Vlasov SDEs remain relatively limited. An important advancement in the classical ($\alpha = 1$) setting is due to Kalinin, Meyer-Brandis, and Proske [17,18], who proved pathwise uniqueness for McKean–Vlasov SDEs with irregular drift using a Yamada–Watanabe-type framework. Their approach relies on stability estimates and regularity assumptions that differ from ours. By contrast, our method applies to the broader fractional setting

$$\alpha > \max\left\{\frac{1}{2} + \frac{1}{p},\ 1 - \frac{1}{2p}\right\}, \quad \text{with } p \geq 2,$$

and explicitly includes the classical case as a special instance (see Remark 3.3 and Corollary 3.7 and Theorem 3.9). Our analysis admits drift and diffusion coefficients with non-Lipschitz and even singular temporal behavior (e.g., via $\eta'(t)/\eta(t)$), and is based on a new auxiliary-function and normalization technique adapted to the smoothing effects of the Caputo derivative (see Example 4.2). This comparison highlights how our framework generalizes previous results to a wider class of McKean–Vlasov SDEs in the $L_p$ setting.

The mathematical foundation of stochastic differential equations is inherently linked to the theory of Brownian motion, which plays a central role in modeling random behavior over time. Brownian motion, also known as the Wiener process, is a continuous-time stochastic process characterized by independent and normally distributed increments with stationary properties. Fundamental contributions to the rigorous development of Brownian motion and its properties include the classical works of Wiener [19], Lévy [20], and Doob [21]. More recent comprehensive treatments can be found in standard texts such as Karatzas and Shreve [22], Øksendal [23], and Revuz and Yor [24], which discuss the stochastic calculus framework, martingale theory, and stochastic integration with respect to Brownian motion. These theoretical underpinnings are crucial for analyzing and solving SDEs, particularly those driven by Brownian motion as considered in this work.

It is well known that pathwise uniqueness is typically ensured under Lipschitz conditions, as shown by Applebaum [25], Ikeda and Watanabe [26], among others (see also [27–29]). However, such conditions are often too restrictive for real-world applications. For example, in fluid dynamics, especially in turbulent regimes, the velocity fields frequently violate Lipschitz continuity (see [30–32]).

The classical Lipschitz condition, while powerful, fails to accommodate the irregularities encountered in many applied problems. To address this, the Osgood condition and the Nagumo-type condition serve as natural generalizations that allow for weaker continuity assumptions while still guaranteeing uniqueness. The Osgood condition, introduced by Osgood in 1898 [33], replaces the Lipschitz constant with a modulus function $G$ satisfying an integral divergence criterion, thus accommodating non-Lipschitz behaviors. On the other hand, the Nagumo-type condition, proposed by Constantin in 2010 [34] as a reinterpretation of Nagumo's uniqueness theorem originally introduced in 1926 [35], controls the growth of the nonlinear term relative to an auxiliary function and an integral inequality. This framework provides a powerful tool for proving uniqueness in settings where traditional methods fail. These conditions have gained increasing attention for their flexibility in handling degenerate or singular behaviors in both deterministic and stochastic systems. In this work, we apply these conditions to the McKean–Vlasov FSDE framework to establish uniqueness under significantly relaxed assumptions.

In contrast to the results obtained by Chu (2018) [36], who established uniqueness for classical ordinary differential equations under the Osgood condition, and Liu & Liu (2024) [37], who employed a combination of Nagumo-type and Perron growth conditions in both ordinary and stochastic settings, our work develops a unified framework for proving pathwise uniqueness of McKean–Vlasov fractional stochastic differential equations in the $L_p$-setting. By extending these classical uniqueness criteria to a fractional stochastic regime with distribution-dependent coefficients, we generalize earlier results and bridge a gap between deterministic analysis and memory-dependent stochastic modeling.

Recent contributions to the fractional McKean–Vlasov framework – such as Kolokoltsov and Troeva [38] and Labed et al. [39] – have primarily focused on weak solutions and distributional formulations driven by fractional Brownian motion. In contrast, our work is, to the best of our knowledge, the first to establish pathwise uniqueness for fractional McKean–Vlasov SDEs with general non-Lipschitz drift and diffusion terms. This is achieved through a novel auxiliary-function and truncation approach that unifies Osgood- and Nagumo-type growth conditions in a distribution-dependent fractional setting within the $L_p$-framework. Our result thus provides a significant extension of classical uniqueness theory in this broader context.

To establish the uniqueness of a continuous solution for an equation, Chu (2018) [36] replaced the standard requirement of Lipschitz continuity for a function $F$ with the following condition:

$$|F(x_2) - F(x_1)| \le G(|x_2 - x_1|), \tag{1.2}$$

where $G$ is a continuous, nondecreasing function with $G(0) = 0, G(r) > 0$ for all $r > 0$, and

$$\lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{G(r)} dr = +\infty.$$

This is known as the *Osgood condition*, originally introduced by Osgood in [33], and represents a significant generalization of the Lipschitz condition. Both the Lipschitz and Osgood conditions are commonly used to guarantee uniqueness in initial-value problems; however, the Lipschitz condition often fails to hold in many real-world scenarios, as previously discussed.

Another important condition for establishing uniqueness is the *Nagumo-type condition*, initially introduced by Constantin in [34] for the first-order differential equation

$$x'(t) = f(t, x),$$

where $T > 0$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(t,0) = 0$ for all $t \in [0,T]$. Constantin showed that, under the initial condition $x(0) = 0$, the equation has only the trivial solution if the following conditions hold:

$$|f(t,x)| \le \frac{u(t)}{u'(t)} \omega(|x|),$$

and

$$\lim_{t \to 0^+} \frac{f(t,x)}{u'(t)} \quad \text{converges uniformly for } |x| \le M,$$

where $u$ is an absolutely continuous function on $[0,T]$ with $u(0) = 0$ and $u'(t)$ defined almost everywhere on $[0,T]$, and $M > 0$ is a given constant. The function $\omega$ belongs to the class

$$\mathcal{F}_M = \left\{ \omega : [0,M] \to [0,\infty) \;\middle|\; \omega \text{ is strictly increasing},\ \omega(0) = 0,\ \int_0^r \frac{\omega(s)}{s} ds \le r,\ \forall r \in (0,M] \right\}.$$

More recently, Chu and Wang (2023) [40] presented a Nagumo-type uniqueness result for a nonlinear differential equation defined on a semi-infinite interval. Subsequently, in 2024, Liu & Liu [37] employed a convex combination of the Nagumo condition and the Perron growth condition to establish the existence of a unique solution for a class of both ordinary and stochastic differential equations.

Throughout this paper, we investigate the pathwise uniqueness of solutions to Eq. (1.1), focusing on cases where the coefficients satisfy the Osgood and Nagumo-type conditions, rather than the classical Lipschitz condition. Our results extend several notable findings already reported in the literature (see, e.g., [37,40–42], and references therein).

An almost surely continuous stochastic process $X : I \times \Omega \to \mathbb{R}^n$ is called a solution to Eq. (1.1) if the following conditions hold:

**(i)** $X(t,\cdot)$ is $\mathcal{F}_t$-measurable for all $t \in I$;

**(ii)**

$$\int_0^t [|(t-s)^{\alpha-1} f(s, X(s,\omega), \mathscr{L}(X(s,\omega)))| + |(t-s)^{\alpha-1} g(s, X(s,\omega), \mathscr{L}(X(s,\omega)))|^2] ds < \infty, \quad \text{a.s.},$$

so that the Riemann integral

$$\int_0^t (t-s)^{\alpha-1} f(s, X(s,\omega), \mathscr{L}(X(s,\omega))) ds$$

and the Itô stochastic integral

$$\int_0^t (t-s)^{\alpha-1} g(s, X(s,\omega), \mathscr{L}(X(s,\omega))) dB(s,\omega)$$

are well-defined;

**(iii)** the process $X(t,\omega)$ satisfies the integral equation

$$\begin{aligned} X(t,\omega) = \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{X_0^{(k)}(\omega)}{k!} t^k &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X(s,\omega), \mathscr{L}(X(s,\omega))) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X(s,\omega), \mathscr{L}(X(s,\omega))) dB(s,\omega), \end{aligned} \tag{1.3}$$

almost surely for any $t \in I$, where $X_0^k(\omega) = \frac{d^k X(t,\omega)}{dt^k}|_{t=0}$ denotes the $k$th derivative of $X(t,\omega)$ at $t = 0$. These are random variables defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$. Furthermore, the process $X(t)$ is assumed to be almost surely differentiable at $t = 0$ up to order $\lfloor \alpha \rfloor$.

The remainder of the paper is organized as follows. In Section 2, we introduce the necessary notations and assumptions, and in Section 3, we present our main result. This result has several implications for both ordinary differential equations and various types of stochastic differential equations, which are discussed in detail. To support the theoretical findings, we present illustrative examples in Section 4, including one in which the drift and diffusion coefficients do not satisfy the standard Lipschitz condition. Finally, the last section summarizes the key contributions of our work and outlines several potential avenues for future research.

## 2. Preliminaries

In this section, we introduce the necessary notations and assumptions that will be used in the subsequent analysis. The norm of a vector is denoted by $|\cdot|$, and the norm of a matrix is denoted by $\|\cdot\|$.

Let $\mathscr{B}(\mathbb{R}^n)$ denote the Borel $\sigma$-algebra on $\mathbb{R}^n$, and let $\mathcal{P}(\mathbb{R}^n)$ represent the set of all probability measures on $\mathscr{B}(\mathbb{R}^n)$, equipped with the topology of weak convergence. Furthermore, define

$$\mathcal{M}_p(\mathbb{R}^n) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^n) \;\middle|\; \|\mu\|_p^p := \int_{\mathbb{R}^n} |x|^p \mu(dx) < \infty \right\}, \quad p \in [2, \infty),$$

which is a Polish space endowed with the $L^p$-Wasserstein distance

$$\mathbb{W}_p(\mu, \nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^p \pi(dx, dy) \right)^{\frac{1}{p}}, \quad \mu, \nu \in \mathcal{M}_p(\mathbb{R}^n),$$

where $\mathscr{C}(\mu, \nu)$ denotes the set of all couplings of $\mu$ and $\nu$.

Note that for any $x \in \mathbb{R}^n$, the Dirac measure $\delta_x$ belongs to $\mathcal{M}_p(\mathbb{R}^n)$ for all $p \in [2, \infty)$, where

$$\delta_x(A) = 1_A(x), \quad \text{for } A \subset \mathbb{R}^n,$$

and $1_A$ is the indicator function of the set $A$, defined as

$$\delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

If $\mu = \mathscr{L}(X)$ and $\nu = \mathscr{L}(Y)$ are the probability laws of the random variables $X$ and $Y$, respectively, then the following inequality holds

$$(\mathbb{W}_p(\mu, \nu))^p \le \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^p \mathscr{L}((X,Y))(dx, dy) = \mathbb{E}(|X-Y|^p) \tag{2.1}$$

where $\mathscr{L}((X,Y))$ denotes the joint distribution of the random vector $(X,Y)$.

### 2.1. Fractional calculus background

We now recall the definitions of fractional operators that are central to our analysis. For consistency with recent works, including Tian and Luo (2023) [43], we adopt the following definitions of fractional calculus operators.

**Definition 2.1.** The Caputo fractional derivative of order $\alpha \in (0,1)$ for a sufficiently smooth function $\varphi$ is given by

$${}^C D_{0+}^{\alpha} \varphi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \varphi'(\tau)\, d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function.

We also recall more general forms of fractional operators, including the Riemann–Liouville integral and the generalized Caputo derivative, which have been employed in recent studies such as [43] to analyze impulsive and delay systems. These operators have also been utilized in a range of significant results in the literature, including our previous works: [44], which investigates Cauchy-type problems for nonlinear fractional differential equations using the measure of noncompactness; [45], which addresses fractional differential equations with functional boundary conditions using inverse operators; and [46], which focuses on fractional evolution equations with non-instantaneous impulses.

**Definition 2.2** (*[43]*)**.** Let $q > 0$ and $\varphi(t)$ be a function defined on $[0,t]$. The Riemann–Liouville fractional integral of order $q$ is given by

$$I_{0+}^{q} \varphi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \varphi(\tau)\, d\tau.$$

**Definition 2.3** (*[43]*)**.** Let $q > 0$ and $m = \lceil q \rceil$. The Caputo derivative of order $q$ is defined as

$${}^C D_{0+}^{q} \varphi(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-\mu)^{m-q-1} \varphi^{(m)}(\mu)\, d\mu,$$

where $m - 1 < q < m$.

For $0 < q < 1$, we also have the identity:

$$I^q_{0+}\,{}^C D^q_{0+}\varphi(t) = \varphi(t) - \varphi(0),$$

which will be useful in our later proofs.

**Remark 2.4.** Suppose $\varphi(t) \in C^m[0,\infty)$ and $m-1 < q < m$, then

$${}^C D^q_t \varphi(t) = \frac{1}{\Gamma(m-q)} \int_0^t \frac{\varphi^{(m)}(s)}{(t-s)^{q+1-m}} ds.$$

As a special case, for any constant function $C(t) \equiv C$, it follows that ${}^C D^q_t C(t) \equiv 0$.

## 3. Main result

The main result of this study is stated in the following theorem, which establishes pathwise uniqueness for the McKean–Vlasov FSDE defined in Eq. (1.1).

**Theorem 3.1.** *Suppose that $\eta : I \to [0,\infty)$ is an absolutely continuous function with $\eta(0) = 0$, and that it has a positive derivative on $(0,T)$ satisfying $\lim_{t\to 0^+} \eta'(t) = +\infty$. Assume that $\phi : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function and*

$$0 < \beta < 2^{-p} T^{1-\frac{p}{2}(2\alpha-1)} \Gamma(\alpha)^p \left( \frac{p-2}{p(2\alpha-1)-2} \right)^{1-\frac{p}{2}},$$

*and let $h : I \to [0,\infty)$ be a continuous function such that the mapping $s \mapsto \frac{h(s)}{\eta(s)}$ is Lebesgue integrable on $I$.*

*Also, let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing, concave function satisfying $\psi(0) = 0$, $\psi(r) > 0$ for all $r > 0$, and*

$$\int_{0^+} \frac{du}{\psi(u)} = +\infty.$$

*Furthermore, let $\alpha > \max\left\{ \frac{1}{2} + \frac{1}{p}, 1 - \frac{1}{2p} \right\}$ and $p \geq 2$, and suppose the continuous functions*

$$f : I \times \mathbb{R}^n \times \mathcal{M}_p(\mathbb{R}^n) \to \mathbb{R}^n, \quad g : I \times \mathbb{R}^n \times \mathcal{M}_p(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$$

*satisfy the following conditions:*

$$|f(t,x,\mu) - f(t,y,\nu)|^p \leq h(t)\psi\left( |x-y|^p + \mathbb{W}^p_p(\mu,\nu) \right), \tag{3.1}$$

$$\|g(t,x,\mu) - g(t,y,\nu)\|^p \leq \beta \frac{\eta'(t)}{\eta(t)} \left( |x-y|^p + \mathbb{W}^p_p(\mu,\nu) \right), \tag{3.2}$$

$$|f(t,x,\mu)|^p \vee \|g(t,x,\mu)\|^p \leq \phi(|x|^p), \tag{3.3}$$

*for all $t \in I$ (in (3.2), $t \in (0,T]$), $x, y \in \mathbb{R}^n$, and $\mu, \nu \in \mathcal{M}_p(\mathbb{R}^n)$.*

*Then, for any random variable $X_0 \in L^p(\Omega;\mathbb{R}^n)$ that is independent of the Brownian motion $\{B(t)\}_{t\in I}$, pathwise uniqueness holds for the solutions to Eq. (1.1).*

**Proof.** Let $X$ and $Y$ be two solutions to the integral form of Eq. (1.1), as specified in Eq. (1.3). Since the $p$th moments of these stochastic processes may not be finite, we employ a truncation method based on stopping times. Specifically, for each integer $k \geq 1$, we define the sequence of indicator functions $I_k(t)$ as

$$I_k(t) = \begin{cases} 1, & \text{if } |X(s)| \leq k \text{ and } |Y(s)| \leq k \text{ for } 0 \leq s \leq t, \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $I_k(t) = I_k(t)I_k(s)$ for any $0 \leq s \leq t \leq T$. Therefore, we obtain the following identity

$$\begin{aligned} I_k(t)[X(t) - Y(t)] = \frac{I_k(t)}{\Gamma(\alpha)} \Bigg\{ &\int_0^t I_k(s)(t-s)^{\alpha-1}[f(s,X(s),\mu_s) - f(s,Y(s),\nu_s)]ds \\ &+ \int_0^t I_k(s)(t-s)^{\alpha-1}[g(s,X(s),\mu_s) - g(s,Y(s),\nu_s)]dB(s) \Bigg\}, \quad t \in I, \end{aligned} \tag{3.4}$$

where $\mu_s = \mathscr{L}_{X(s)}$ and $\nu_s = \mathscr{L}_{Y(s)}$ for $0 \leq s \leq t$. By assumption (3.3) and Jensen's inequality (noting that $p \geq 2$), we obtain

$$\left| I_k(t)[f(t,X(t),\mu_t) - f(t,Y(t),\nu_t)] \right|^p \leq 2^{p-1} I^p_k(t)\left( |f(t,X(t),\mu_t)|^p + |f(t,Y(t),\nu_t)|^p \right) \leq 2^p \phi(k^p), \tag{3.5}$$

for all $t \in I$. Similarly, we have

$$\left| I_k(t)[g(t,X(t),\mu_t) - g(t,Y(t),\nu_t)] \right|^p \leq 2^p \phi(k^p), \quad t \in I. \tag{3.6}$$

Hence, the $p$th moments of both integrals in (3.4) exist, and we obtain the estimate

$$\begin{aligned}&\int_0^t (t-s)^{(\alpha-1)p}\Big|I_k(s)[f(s,X(s),\mu_s)-f(s,Y(s),\nu_s)]\Big|^p ds\\&+\int_0^t (t-s)^{(\alpha-1)p}\Big|I_k(s)[g(s,X(s),\mu_s)-g(s,Y(s),\nu_s)]\Big|^p dB(s)\\&\le 2^p\phi(k^p)\left(\frac{T^{(\alpha-1)p+1}}{(\alpha-1)p+1}+M\right),\end{aligned}$$

where $M$ is an upper bound for $|\int_0^t (t-s)^{(\alpha-1)p}dB(s)|$.

To justify the existence of such an upper bound, we apply Itô's isometry

$$\mathbb{E}\left[\left(\int_0^t (t-s)^{(\alpha-1)p}dB(s)\right)^2\right]=\int_0^t (t-s)^{2(\alpha-1)p}ds\le\frac{T^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}<\infty,$$

This shows that the second moment (variance) of the integral is finite, thus ensuring that a suitable bound $M$ exists.

Since $I_k(t)\in\{0,1\}$, and by applying Jensen's inequality along with Burkholder–Davis–Gundy's inequality to (3.4), we obtain

$$\begin{aligned}\mathbb{E}\Big(I_k(t)|X(t)-Y(t)|^p\Big)&\le\frac{2^{p-1}}{\Gamma(\alpha)^p}\mathbb{E}\left|\int_0^t I_k(s)(t-s)^{\alpha-1}[f(s,X(s),\mu_s)-f(s,Y(s),\nu_s)]ds\right|^p\\&+\frac{2^{p-1}}{\Gamma(\alpha)^p}\mathbb{E}\left|\int_0^t I_k(s)(t-s)^{\alpha-1}[g(s,X(s),\mu_s)-g(s,Y(s),\nu_s)]dB(s)\right|^p\\&\le\frac{2^{p-1}t^{\alpha p-1}}{\Gamma(\alpha)^p}\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\int_0^t\mathbb{E}\Big(I_k(s)|f(s,X(s),\mu_s)-f(s,Y(s),\nu_s)|^p\Big)ds\\&+\frac{2^{p-1}}{\Gamma(\alpha)^p}\left(\int_0^t (t-s)^{2(\alpha-1)}\mathbb{E}\Big(I_k(s)|g(s,X(s),\mu_s)-g(s,Y(s),\nu_s)|^2\Big)ds\right)^{\frac{p}{2}}\\&\le\frac{2^{p-1}t^{\alpha p-1}}{\Gamma(\alpha)^p}\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\int_0^t\mathbb{E}\Big(I_k(s)|f(s,X(s),\mu_s)-f(s,Y(s),\nu_s)|^p\Big)ds\\&+\frac{2^{p-1}t^{\frac{p}{2}(2\alpha-1)-1}}{\Gamma(\alpha)^p}\left(\frac{p-2}{p(2\alpha-1)-2}\right)^{\frac{p}{2}-1}\int_0^t\mathbb{E}\Big(I_k(s)|g(s,X(s),\mu_s)-g(s,Y(s),\nu_s)|^p\Big)ds.\end{aligned}\tag{3.7}$$

From a different perspective, it follows directly from (3.1) that

$$I_k(t)|f(t,X(t),\mu_t)-f(t,Y(t),\nu_t)|^p\le h(t)\psi(I_k(t)|X(t)-Y(t)|^p+I_k(t)\mathbb{W}_p^p(\mu_t,\nu_t)),\quad t\in I.\tag{3.8}$$

Similarly, condition (3.2) ensures that

$$I_k(t)|g(t,X(t),\mu_t)-g(t,Y(t),\nu_t)|^p\le\beta\frac{\eta'(t)}{\eta(t)}I_k(t)(|X(t)-Y(t)|^p+\mathbb{W}_p^p(\mu_t,\nu_t)),\quad t\in(0,T].\tag{3.9}$$

Now, applying Jensen's inequality to the concave function $\psi$, we deduce

$$\mathbb{E}\psi\Big(I_k(t)(|X(t)-Y(t)|^p+\mathbb{W}_p^p(\mu_t,\nu_t))\Big)\le\psi\Big(\mathbb{E}(I_k(t)(|X(t)-Y(t)|^p+\mathbb{W}_p^p(\mu_t,\nu_t)))\Big)\quad t\in I.\tag{3.10}$$

Define the continuous function $u_k: I\to[0,(2k)^p]$ by

$$u_k(t)=\mathbb{E}\big(I_k(t)|X(t)-Y(t)|^p\big),\quad t\in I.\tag{3.11}$$

Combining Eqs. (2.1), (3.7)–(3.10), we arrive at the following key inequality

$$\begin{aligned}u_k(t)\le&\frac{2^{p-1}t^{\alpha p-1}}{\Gamma(\alpha)^p}\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\int_0^t h(s)\psi(2u_k(s))ds\\&+\frac{2^p t^{\frac{p}{2}(2\alpha-1)-1}\beta}{\Gamma(\alpha)^p}\left(\frac{p-2}{p(2\alpha-1)-2}\right)^{\frac{p}{2}-1}\int_0^t\frac{\eta'(s)}{\eta(s)}u_k(s)ds,\quad t\in(0,T].\end{aligned}\tag{3.12}$$

The first integral on the right-hand side of (3.12), along with inequality (3.8), stems directly from applying the Nagumo-type condition (inequality (3.1)), which bounds the growth of the drift coefficient $f$ using the sublinear function $\psi$. This control plays a key role in recursively estimating the difference $u_k(t)=\mathbb{E}(I_k(t)|X(t)-Y(t)|^p)$.

We now claim that

$$\lim_{t\to0^+}\frac{u_k(t)}{\eta(t)}=0,\quad k=1,2,\ldots\tag{3.13}$$

To verify this claim, we observe that the assumption $\lim_{t\to0^+}\eta'(t)=+\infty$ implies that for any $\epsilon>0$, there exists $\delta=\delta(k,\epsilon)>0$ such that

$$2^p\phi(k^p)\le\epsilon\eta'(t),\quad t\in(0,\delta].$$

Using this bound in inequalities (3.5)–(3.7), we obtain

$$\begin{aligned} u_k(t) &\le \frac{2^{p-1}t^{\alpha p-1}}{\Gamma(\alpha)^p}\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\epsilon\int_0^t \eta'(s)ds \\ &\quad+\frac{2^{p-1}t^{\frac{p}{2}(2\alpha-1)-1}}{\Gamma(\alpha)^p}\left(\frac{p-2}{p(2\alpha-1)-2}\right)^{\frac{p}{2}-1}\epsilon\int_0^t \eta'(s)ds \\ &= \frac{2^{p-1}\epsilon t^{\frac{p}{2}(2\alpha-1)-1}}{\Gamma(\alpha)^p}\left[t^{\frac{p}{2}}\left(\frac{p-1}{\alpha p-1}\right)^{p-1}+\left(\frac{p-2}{p(2\alpha-1)-2}\right)^{\frac{p}{2}-1}\right]\eta(t) \\ &=: \epsilon t^{\frac{p}{2}(2\alpha-1)-1}(M_p t^{\frac{p}{2}}+N_p)\eta(t), \quad t\in(0,\delta], \end{aligned}$$

which establishes the validity of (3.13).

Considering the continuity of the functions $u_k$ and $\eta$, and in light of (3.13), we define the function

$$v_k(t)=\frac{u_k(t)}{\eta(t)}, \quad t\in(0,T], \tag{3.14}$$

which can be continuously extended to the closed interval $I$, with $v_k(0):=0$.

Next, we introduce a continuous, nondecreasing function $w_k : I\to[0,\infty)$, defined by

$$w_k(t)=\max_{0\le s\le t}\{v_k(s)\}, \quad t\in I. \tag{3.15}$$

Let us set $\lambda := 2\eta(T)>0$. Referring back to inequality (3.12), we obtain

$$\begin{aligned} v_k(t) &\le \frac{M_p t^{\alpha p-1}}{\eta(t)}\int_0^t h(s)\psi(2u_k(s))ds+\frac{2\beta N_p t^{\frac{p}{2}(2\alpha-1)-1}}{\eta(t)}\int_0^t \frac{\eta'(s)}{\eta(s)}u_k(s)ds \\ &= \frac{M_p t^{\alpha p-1}}{\eta(t)}\int_0^t h(s)\psi(2v_k(s)\eta(s))ds+\frac{2\beta N_p t^{\frac{p}{2}(2\alpha-1)-1}}{\eta(t)}\int_0^t \eta'(s)v_k(s)ds \\ &\le M_p t^{\alpha p-1}\int_0^t \frac{h(s)}{\eta(s)}\psi(\lambda v_k(s))ds+\frac{2\beta N_p t^{\frac{p}{2}(2\alpha-1)-1}}{\eta(t)}\int_0^t \eta'(s)v_k(s)ds, \quad t\in(0,T], \end{aligned} \tag{3.16}$$

where the last inequality holds due to the monotonicity of $\eta$ and $\psi$.

Since $v_k(s)\le w_k(t)$ for all $0\le s\le t\le T$ by definition (3.15), it follows that

$$\begin{aligned} v_k(t) &\le M_p t^{\alpha p-1}\int_0^t \frac{h(s)}{\eta(s)}\psi(\lambda v_k(s))ds+\frac{2\beta N_p t^{\frac{p}{2}(2\alpha-1)-1}}{\eta(t)}\int_0^t \eta'(s)v_k(s)ds \\ &\le M_p T^{\alpha p-1}\int_0^t \frac{h(s)}{\eta(s)}\psi(\lambda w_k(s))ds+2\beta N_p T^{\frac{p}{2}(2\alpha-1)-1}w_k(t), \quad t\in(0,T], \end{aligned}$$

where we used the monotonicity of $\psi$.

Because the right-hand side is increasing with respect to $t$, taking the supremum on both sides and recalling (3.15), we conclude

$$w_k(t)\le \frac{M_p T^{\alpha p-1}}{1-2\beta N_p T^{\frac{p}{2}(2\alpha-1)-1}}\int_0^t \frac{h(s)}{\eta(s)}\psi(\lambda w_k(s))ds, \quad t\in(0,T]. \tag{3.17}$$

To simplify further analysis, we define a new continuous, nondecreasing function $z_k : I\to[0,\infty)$ as

$$z_k : I\to[0,\infty), \quad z_k(t)=\lambda w_k(t), \quad t\in I. \tag{3.18}$$

Using this substitution, inequality (3.17) becomes

$$z_k(t)\le\int_0^t \varphi(s)\psi(z_k(s))ds, \quad t\in(0,T], \tag{3.19}$$

where the function $\varphi : (0,T]\to[0,\infty)$ is defined by

$$\varphi : (0,T]\to[0,\infty), \quad \varphi(t)=\frac{\lambda M_p T^{\alpha p-1}}{1-2\beta N_p T^{\frac{p}{2}(2\alpha-1)-1}}\cdot\frac{h(t)}{\eta(t)}, \quad t\in(0,T],$$

and satisfies $\varphi\in L^1(I)$ by assumption.

**Claim:** Given that $z_k(0)=0$, we assert that (3.19) necessitates $z_k(t)=0$ for all $t\in I$. Consequently, it follows from (3.14), (3.15), and (3.18) that

$$u_k(t)=0, \quad t\in I. \tag{3.20}$$

To substantiate this assertion, we proceed by contradiction. Assume that there exists some $t_0\in(0,T]$ such that $z_k(t_0)>0$. Since $z_k$ is continuous and nondecreasing, define

$$\tau=\inf\{t\in[0,t_0] : z_k(t)>0\}.$$

It then follows that $z_k(t) = 0$ for all $t \in [0, \tau)$, and $z_k(t) > 0$ for all $t \in (\tau, T]$.

Therefore, inequality (3.19) implies

$$0 < z_k(t) \leq \int_\tau^t \varphi(s)\psi(z_k(s))ds, \quad t \in [\tau, T]. \tag{3.21}$$

Define the auxiliary function

$$U_k(t) = \int_\tau^t \varphi(s)\psi(z_k(s))ds, \quad t \in [\tau, T].$$

This function is differentiable over $(\tau, T)$, and its derivative satisfies

$$U_k'(t) = \varphi(t)\psi(z_k(t)) \leq \varphi(t)\psi(U_k(t)), \quad t \in (\tau, T).$$

where the inequality follows from (3.21) and the monotonicity of $\psi$.

Since $U_k(t) > 0$ for all $t \in (\tau, T)$, we may write

$$\frac{U_k'(t)}{\psi(U_k(t))} \leq \varphi(t), \quad t \in (\tau, T).$$

Integrating both sides over the interval $[t, t_1]$, for $\tau < t < t_1 < T$, we get

$$\int_{U_k(t)}^{U_k(t_1)} \frac{dr}{\psi(r)} \leq \int_t^{t_1} \varphi(s)ds. \tag{3.22}$$

Taking the limit as $t \to \tau^+$ and noting that $U_k(\tau) = 0$, we find

$$\lim_{t\to\tau^+} \int_{U_k(t)}^{U_k(t_1)} \frac{dr}{\psi(r)} \leq \int_\tau^{t_1} \varphi(s)ds \leq \|\varphi\|_{L^1(I)} < \infty. \tag{3.23}$$

At this stage, we invoke the Osgood condition, namely $\int_0^\delta \frac{dr}{\psi(r)} = +\infty$, imposed on the control function $\psi$. This condition implies that the left-hand side of (3.22) must diverge as $t \to \tau^+$, while the right-hand side remains finite (as shown in (3.23)). This contradiction implies that our assumption $z_k(t) > 0$ must be false. Therefore, we conclude that $z_k(t) \equiv 0$, completing the argument and confirming the validity of (3.20).

From (3.11) and (3.20), we conclude that

$$I_k(t)X(t) = I_k(t)Y(t), \quad \text{a.s. for all } t \in I.$$

Since both processes $X(t)$ and $Y(t)$ are continuous (and hence bounded on compact intervals), we can make the probability of the complement of the event $I_k(t) = 1$ arbitrarily small by choosing a sufficiently large constant $M \geq 1$, as follows

$$P(I_k(t) \neq 1 \text{ on } I) \leq P(\sup_{t\in I}\{|X(t)|\} > M) + P(\sup_{t\in I}\{|Y(t)|\} > M).$$

Hence, for any fixed $t \in I$, we deduce that $X(t) = Y(t)$ almost surely. As this equality holds on a countable dense subset $C \subset I$, and since both processes are a.s. continuous, it follows that they must coincide on the entire interval $I$.

Finally, since the event $\{X(t) \neq Y(t)\}$ has probability zero for each $t \in I$, we conclude

$$P\left(\sup_{t\in I} |X(t) - Y(t)| > 0\right) = 0.$$

This establishes the claim and completes the proof. □

**Remark 3.2.** The condition

$$\alpha > \max\left\{\frac{1}{2} + \frac{1}{p},\ 1 - \frac{1}{2p}\right\}$$

plays a key role in ensuring that both the deterministic and stochastic estimates in the proof remain integrable near the origin. Specifically, the term $t^{\alpha p-1}$, which arises when estimating the deterministic convolution integral, must be integrable over $(0, T]$ and must decay as $t \to 0^+$. This is ensured when $\alpha p - 1 > 0$, or equivalently, $\alpha > \frac{1}{p}$. However, for the more delicate argument involving the decay $u_k(t)/\eta(t) \to 0$, we require a stronger condition, namely $\alpha > 1 - \frac{1}{2p}$, as shown below. More precisely, this condition appears explicitly in the stochastic estimate via Itô's isometry:

$$\mathbb{E}\left[\left(\int_0^t (t-s)^{(\alpha-1)p}\, dB(s)\right)^2\right] = \int_0^t (t-s)^{2(\alpha-1)p} ds.$$

The above integral is finite if and only if $2(\alpha - 1)p > -1$, which leads to the condition $\alpha > 1 - \frac{1}{2p}$.

In parallel, the time-singular prefactor $t^{\frac{p}{2}(2\alpha-1)-1}$ appears in the stochastic estimates obtained via the Burkholder–Davis–Gundy inequality. To ensure decay near zero and integrability, we require $\frac{p}{2}(2\alpha - 1) - 1 > 0$, which yields $\alpha > \frac{1}{2} + \frac{1}{p}$.

Hence, the stated lower bound on $\alpha$ guarantees the required integrability and regularity of both deterministic and stochastic terms, which is essential for establishing the uniqueness result.

**Special case:** When $g \equiv 0$ (i.e., no diffusion term), the equation reduces to a deterministic Caputo-type fractional differential equation

$$^{C}D^{\alpha}X(t) = f(t, X(t), \mathscr{L}_{X(t)}).$$

In this deterministic setting, the stochastic integral vanishes, and so does the corresponding singular prefactor $t^{\frac{p}{2}(2\alpha-1)-1}$ from the stochastic estimate. Therefore, the condition $\alpha > 1 - \frac{1}{2p}$, which arises from Itô's isometry and is necessary to control the second moment of the stochastic term, is no longer needed. The only integrability requirement for the deterministic part comes from ensuring

$$\int_0^t (t-s)^{\alpha p-1}\,ds < \infty,$$

which holds as long as $\alpha > 0$. Hence, in the deterministic case ($g \equiv 0$), the condition on $\alpha$ can be relaxed to $\alpha > 0$, significantly broadening the admissible range of the fractional order.

**Remark 3.3.** Building upon the reasoning presented above, it becomes evident that Theorem 3.1 also holds in the case where $\alpha = 1$, which leads directly to Corollaries 3.8 and 3.10.

**Remark 3.4.** The proof of Theorem 3.1 suggests that a comparable result can be obtained by applying a Nagumo-type condition to the diffusion term and an Osgood-type condition to the drift term. Reproducing the core arguments yields the same conclusion. We briefly sketch the idea below.

In this reversed setting, we assume the drift $f$ satisfies an inequality of the form

$$|f(t,x,\mu) - f(t,y,\nu)|^p \le \beta\,\frac{\eta'(t)}{\eta(t)}\left(|x-y|^p + \mathbb{W}_p^p(\mu,\nu)\right),$$

while the diffusion $g$ satisfies the sublinear growth bound

$$\|g(t,x,\mu) - g(t,y,\nu)\|^p \le h(t)\,\psi\left(|x-y|^p + \mathbb{W}_p^p(\mu,\nu)\right),$$

where $\psi$ is concave, nondecreasing, and satisfies the Osgood condition $\int_{0^+} \frac{du}{\psi(u)} = +\infty$.

Following the same steps as in the original proof, one derives a differential inequality for the error term $u_k(t) = \mathbb{E}[I_k(t)|X(t) - Y(t)|^p]$, leading to

$$u_k(t) \le A(t)\int_0^t \frac{\eta'(s)}{\eta(s)} u_k(s)\,ds + B(t)\int_0^t h(s)\,\psi(2u_k(s))\,ds,$$

for some functions $A(t)$ and $B(t)$ defined on the interval $I$. This inequality is structurally identical to the original estimate given in the proof of Theorem 3.1. Defining $v_k(t) = \frac{u_k(t)}{\eta(t)}$, and repeating the comparison argument based on the Osgood condition, one concludes that $u_k(t) \equiv 0$, hence establishing pathwise uniqueness. Thus, uniqueness holds under the reversed roles of the Osgood and Nagumo conditions.

As a technical note, one may observe that in this reversed setting, the singular coefficient $\frac{\eta'(t)}{\eta(t)}$ appears in the drift term rather than in the stochastic estimate. However, the convolution integral involving the drift remains integrable due to the smoothing effect of the fractional kernel $(t-s)^{\alpha-1}$. Moreover, the normalization by $\eta(t)$ continues to ensure decay of the error term near zero. Consequently, the singular behavior remains controlled, and the uniqueness proof proceeds without obstruction.

Importantly, the condition $\alpha > \max\left\{\frac{1}{2} + \frac{1}{p},\ 1 - \frac{1}{2p}\right\}$ remains necessary in this reversed setting to ensure the integrability of both the stochastic and deterministic components of the proof, as well as the appropriate decay of certain terms.

**Remark 3.5** (*Extension to Lévy-Driven Systems*)**.** The uniqueness result in Theorem 3.1 can be extended to the case where the Brownian motion $B(t)$ is replaced by a Lévy process $L(t)$, under suitable integrability conditions. Suppose $L(t)$ admits the Lévy–Itô decomposition with both a Brownian component and a jump component, and satisfies the finite $p$th moment condition:

$$\int_{\mathbb{R}^d} \left(|z|^2 \wedge |z|^p\right) \nu(dz) < \infty,$$

where $\nu$ is the Lévy measure. Then, the fractional McKean–Vlasov equation takes the form:

$$X(t) = \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{X_0^{(k)}}{k!} t^k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, X(s), \mathscr{L}_{X(s)})\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, X(s), \mathscr{L}_{X(s)})\,dL(s),$$

where the last term is interpreted as an Itô integral with respect to Lévy noise.

The core structure of the uniqueness proof remains intact: the truncation argument, the auxiliary-function normalization, and the $L_p$-based estimates continue to apply due to the smoothing nature of the fractional kernel. While the presence of jumps introduces path discontinuities, these are handled using generalized Burkholder–Davis–Gundy or Kunita-type inequalities adapted for Lévy integrals (see, e.g., [25,26]).

Importantly, the assumptions (3.1)–(3.3) on the drift and diffusion coefficients do not require modification, and the singular behavior near $t = 0$ remains controlled by the same mechanisms. Thus, Theorem 3.1 remains valid under Lévy noise, provided the Lévy process has finite $p$th moments.

A complete extension to more general classes of Lévy noise (including infinite activity and pure-jump Lévy processes) will be the subject of a forthcoming study.

As an immediate consequence of Theorem 3.1, we derive an extension of the classical uniqueness result for Lipschitz continuous functions – now including a distribution-dependent term – viewed here as a particular case in the $L^2$ setting with $\alpha \geq 1$.

**Theorem 3.6.** *Assume that $\alpha \geq 1$. Suppose there exists a constant $C > 0$ such that the continuous functions $f : I \times \mathbb{R}^n \times \mathcal{M}_2(\mathbb{R}^n) \to \mathbb{R}^n$, $g : I \times \mathbb{R}^n \times \mathcal{M}_2(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$ satisfy the following conditions:*

$$|f(t,x,\mu) - f(t,y,\nu)|^2 \vee \|g(t,x,\mu) - g(t,y,\nu)\|^2 \leq C\left(|x-y|^2 + \mathbb{W}_2^2(\mu,\nu)\right), \quad t \in I,\ x,y \in \mathbb{R}^n,\ \mu,\nu \in \mathcal{M}_2(\mathbb{R}^n),$$

$$|f(t,x,\mu)|^2 \vee \|g(t,x,\mu)\|^2 \leq C^2(1+|x|^2), \quad (t,x,\mu) \in I \times \mathbb{R}^n \times \mathcal{M}_2(\mathbb{R}^n).$$

*Then, for any random variable $X_0 \in L^2(\Omega,\mathbb{R}^n)$, independent of the Brownian motion $\{B(t)\}_{t\in I}$, pathwise uniqueness holds for solutions to Eq. (1.1).*

**Proof.** Let us define $\psi(x) = x$, and choose $0 < \beta < \frac{1}{4}T^{2(1-\alpha)}\Gamma(\alpha)^2$, $h(t) = C$, $\phi(t) = C^2(1+x)$, and

$$\eta(t) = t^p \exp(rt^q),$$

where $p,q > 0$, $\max\{p,q\} < 1$, and

$$r = \frac{C^2T^{1-q}}{\beta q}.$$

It is worth noting that in defining $\eta(t)$, the choice $q \geq 1$ is still feasible, provided a lower bound is imposed on $p < 1$, particularly when $C \in \left(0, \frac{1}{2}T^{\frac{1}{2}-\alpha}\right)$.

Under these assumptions, the desired result follows directly from Theorem 3.1. □

**Corollary 3.7.** *Assume $\alpha \geq 1$ is given. Suppose there exists a constant $C > 0$ such that the continuous functions $f : I \times \mathbb{R}^n \times \mathcal{M}_2(\mathbb{R}^n) \to \mathbb{R}^n$, $g : I \times \mathbb{R}^n \times \mathcal{M}_2(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$ satisfy the following conditions:*

$$|f(t,x,\mu) - f(t,y,\nu)|^2 \leq C\left(|x-y|^2 + \mathbb{W}_2^2(\mu,\nu)\right),$$
$$\|g(t,x,\mu) - g(t,y,\nu)\|^2 \leq \lambda t^r\left(|x-y|^2 + \mathbb{W}_2^2(\mu,\nu)\right),$$

*for all $t \in I$, $x,y \in \mathbb{R}^n$, and $\mu,\nu \in \mathcal{M}_2(\mathbb{R}^n)$, where $r \geq -1$, and*

$$0 < \lambda < \frac{1}{4}T^{1-2\alpha-r}\Gamma(\alpha)^2.$$

*Also assume that*

$$|f(t,x,\mu)|^2 \vee \|g(t,x,\mu)\|^2 \leq C^2(1+|x|^2), \quad (t,x,\mu) \in I \times \mathbb{R}^n \times \mathcal{M}_2(\mathbb{R}^n).$$

*Then, for any random variable $X_0 \in L^2(\Omega,\mathbb{R}^n)$, independent of the Brownian motion $\{B(t)\}_{t\in I}$, the solution to Eq. (1.1) is pathwise unique.*

**Proof.** We apply Theorem 3.1 in the particular case where $\psi(x) = x$, $h(t) = C$, and $\eta(t) = t^s$, with

$$\frac{\lambda T^{r+1}}{\beta} \leq s < 1 \quad \text{and} \quad \beta \in \left(\lambda T^{r+1}, \frac{1}{4}T^{2(1-\alpha)}\Gamma(\alpha)^2\right).$$

Under these assumptions, the conditions of Theorem 3.1 are satisfied, and the result follows. □

As a direct consequence of Theorem 3.1, we can now derive several results concerning different classes of stochastic differential equations (SDEs).

### *3.1. Classical stochastic McKean–Vlasov differential equation*

By setting $\alpha = 1$ in conditions (i)–(iii) of the previous section, we can adapt the hypotheses of Theorem 3.1 to the classical stochastic McKean–Vlasov differential equation, which is given by

$$dX(t) = f(t,X(t),\mathscr{L}_{X(t)})dt + g(t,X(t),\mathscr{L}_{X(t)})dB(t), \quad t \in (0,T], \tag{3.24}$$

where $X(0) = X_0$, and all notations remain as previously defined.

**Corollary 3.8.** *In Theorem 3.1, let us suppose $\beta \in (0, 2^{-p}T^{1-\frac{p}{2}})$, $p \geq 2$, and remove the condition on $\alpha$. Assuming all other conditions are satisfied, then for a given random variable $X_0 \in L^p(\Omega,\mathbb{R}^n)$ that is independent of the Brownian motion $\{B(t)\}_{t\in I}$, pathwise uniqueness is guaranteed for the solutions to Eq. (3.24).*

### 3.2. Pathwise uniqueness of the solutions to nonlinear FSDEs

Consider the nonlinear fractional stochastic differential equation

$$ {}^{C}D_0^{\alpha} X(t) = f(t, X(t)) + g(t, X(t))\frac{dB(t)}{dt}, \quad t \in (0, T], \tag{3.25} $$

with initial condition $X(0) = X_0$, and notations as previously defined. Theorem 3.1 leads to the following result for nonlinear fractional stochastic differential equations (FSDEs) that do not include a distribution term. In this setting, the admissible range for $\beta$ effectively doubles, and the Wasserstein distance terms on the right-hand sides of Eqs. (3.1) and (3.2) are omitted.

We now formulate a theorem in which the regularity assumptions on the drift and diffusion terms are expressed using generalized control functions $\psi$ and $\eta$. Concrete examples of such functions – for instance, a piecewise-defined $\psi$ with logarithmic singularity, and $\eta(t) = t^{\gamma}$ with $\gamma \in (0, 1)$ – are presented in Example 4.2, which demonstrates the validity of these conditions in a specific nonlinear FSDE.

**Theorem 3.9.** *In Theorem 3.1, assume that*

$$ 0 < \beta < 2^{1-p} T^{1-\frac{p}{2}(2\alpha-1)} \Gamma(\alpha)^p \left( \frac{p-2}{p(2\alpha-1)-2} \right)^{1-\frac{p}{2}}, \quad p \geq 2, $$

*and that the remaining notations are defined similarly to those in Theorem 3.1. Assume further that the continuous functions $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $g : I \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the following conditions:*

$$ |f(t, x) - f(t, y)|^p \leq h(t)\psi(|x - y|^p), \quad t \in I, \ x, y \in \mathbb{R}^n, \tag{3.26} $$

$$ \|g(t, x) - g(t, y)\|^p \leq \beta \frac{\eta'(t)}{\eta(t)} |x - y|^p, \quad t \in (0, T], \ x, y \in \mathbb{R}^n, \tag{3.27} $$

$$ |f(t, x)|^p \vee \|g(t, x)\|^p \leq \phi(|x|^p), \quad (t, x) \in I \times \mathbb{R}^n. \tag{3.28} $$

*Then, for a given random variable $X_0 \in L^p(\Omega, \mathbb{R}^n)$ that is independent of the Brownian motion $\{B(t)\}_{t\in I}$, the fractional stochastic differential Eq. (3.25) admits a pathwise unique solution. Furthermore, the sequence of successive approximations:*

$$ X_{k+1}(t) = \sum_{l=0}^{\lfloor\alpha\rfloor} \frac{X_0^{(l)}}{l!} t^l + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, X_k(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s, X_k(s))\, dB(s), \tag{3.29} $$

*with $X_0(t) = X_0$, converges uniformly to this unique solution on $I$.*

**Proof.** To establish the first statement of Theorem 3.9, we replicate the proof of Theorem 3.1 step by step. The only modifications required are: removing the term $\mathbb{W}_p^p(\mu_t, \nu_t)$ throughout the proof, defining $\lambda := \eta(T)$, and replacing $u_k(s)$ with $\frac{1}{2}u_k(s)$ on the right-hand sides of Eqs. (3.12) and (3.16). These replacements should be consistently applied throughout the remainder of the proof.

To demonstrate the second statement, note that Eq. (3.25) has a unique solution on the interval $I$. We now prove the uniform convergence of the sequence of successive approximations defined in Eq. (3.29) to this unique solution over $I$.

Define the scalar continuous function $u_k(t) := |X_{k+1}(t) - X_k(t)|^p$. We claim that $u_k(t) \to 0$ uniformly on $I$. This can be shown using the reasoning from Eq. (3.11), adapted for the present case, leading to the estimate

$$ \begin{aligned} u_k(t) &\leq \frac{2^{p-1}}{\Gamma(\alpha)^p} \left| \int_0^t (t - s)^{\alpha-1} (f(s, X_k(s)) - f(s, X_{k-1}(s))) ds \right|^p \\ &\quad + \frac{2^{p-1}}{\Gamma(\alpha)^p} \left| \int_0^t (t - s)^{\alpha-1} (g(s, X_k(s)) - g(s, X_{k-1}(s))) dB(s) \right|^p \\ &\leq M_p t^{\alpha p-1} \int_0^t h(s)\psi(u_{k-1}(s)) ds + N_p t^{\frac{p}{2}(2\alpha-1)-1} \int_0^t \frac{\eta'(s)}{\eta(s)} u_{k-1}(s) ds, \qquad t \in I, \quad k \geq 1, \end{aligned} $$

where

$$ M_p := \frac{2^{p-1}}{\Gamma(\alpha)^p} \left( \frac{p-1}{\alpha p - 1} \right)^{p-1}, \quad N_p := \frac{2^{p-1}\beta}{\Gamma(\alpha)^p} \left( \frac{p-2}{p(2\alpha-1)-2} \right)^{\frac{p}{2}-1}. $$

Following the steps outlined in Eqs. (3.11)–(3.23), we conclude that the sequence $\{X_k(t)\}_{k\geq 0}$ is contained in the ball $B_M(X_0) := \{X \in \mathbb{R}^n \mid |X - X_0| \leq M\}$ for some $M > 0$.

Moreover, from Eq. (3.29), we immediately observe that the sequence $\{X_k(t)\}_{k\geq 1}$ is equicontinuous and uniformly bounded over $I$. This is verified using the Doob martingale inequality and Itô's formula:

$$ \begin{aligned} |X_{k+1}(t) - X_0| &\leq \left| \sum_{l=1}^{\lfloor\alpha\rfloor} \frac{X_0^{(l)}}{l!} t^l \right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s, X_k(s))| ds + \frac{1}{\Gamma(\alpha)} \left( \int_0^t (t - s)^{2(\alpha-1)} \|g(s, X_k(s))\|^2 ds \right)^{1/2} \\ &\leq \sup_{s\in[0,T]} \left| \sum_{n=1}^{\lfloor\alpha\rfloor} \frac{X_0^{(n)}}{n!} s^n \right| + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \sup_{\substack{s\in[0,T] \\ x\in B_M(x_0)}} |f(s, x)| + \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \sup_{\substack{s\in[0,T] \\ x\in B_M(x_0)}} \|g(s, x)\| < \infty, \quad \text{for all } t \in I, \end{aligned} $$

which confirms that the sequence is uniformly bounded on $I$. We note that the term $\left|\sum_{l=1}^{\lfloor\alpha\rfloor}\frac{X_0^{(l)}}{l!}t^l\right|$ is included only when $\alpha \geq 1$; otherwise, it is omitted.

Next, to establish equicontinuity, we consider the estimate

$$\begin{aligned}|X_{k+1}(t)-X_{k+1}(s)| &\leq \sum_{l=1}^{\lfloor\alpha\rfloor}\frac{X_0^{(l)}}{l!}|t^l-s^l| + \frac{1}{\Gamma(\alpha)}\int_0^s |(t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1}|\cdot|f(\tau,X_k(\tau))|d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\left(\int_0^s ((t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1})^2\|g(\tau,X_k(\tau))\|^2 d\tau\right)^{\frac{1}{2}} \\ &+ \frac{1}{\Gamma(\alpha)}\int_s^t (t-\tau)^{\alpha-1}|f(\tau,X_k(\tau))|d\tau + \frac{1}{\Gamma(\alpha)}\left(\int_s^t (t-\tau)^{2(\alpha-1)}\|g(\tau,X_k(\tau))\|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq \sum_{l=1}^{\lfloor\alpha\rfloor}\frac{X_0^{(l)}}{(l-1)!}\max\{t,s\}^{l-1}|t-s| + I_1+I_2+I_3+I_4, \quad 0\leq s<t\leq T.\end{aligned} \tag{3.30}$$

For convenience, we denote the integral terms by $I_1, I_2, I_3, I_4$. The estimates for $I_1, I_3$, and $I_4$ are as follows:

$$\begin{aligned} I_1 &= \frac{|(t-s)^\alpha-(t^\alpha-s^\alpha)|}{\Gamma(\alpha+1)}\sup_{\substack{\tau\in[0,T]\\ x\in B_M(x_0)}}\{|f(\tau,x)|\}, \quad I_3 = \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\sup_{\substack{\tau\in[0,T]\\ x\in B_M(x_0)}}\{|f(\tau,x)|\}, \\ I_4 &= \frac{(t-s)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\sup_{\substack{\tau\in[0,T]\\ x\in B_M(x_0)}}\{\|g(\tau,x)\|\}, \quad 0\leq s<t\leq T.\end{aligned} \tag{3.31}$$

Next, to handle $I_2$, we introduce the substitution $u=\frac{s-\tau}{t-\tau}$ under the assumption $0\leq\tau\leq s<t\leq T$, which yields

$$\begin{aligned} I_2 &= \frac{1}{\Gamma(\alpha)}\left(\int_0^s ((t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1})^2\|g(\tau,X_k(\tau))\|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq \frac{(t-s)^{2\alpha-1}}{\Gamma(\alpha)}\left(\int_0^{s/t}\frac{(1-u^{\alpha-1})^2}{(1-u)^{2\alpha}}du\right)\sup_{\substack{\tau\in[0,T]\\ x\in B_M(x_0)}}\{\|g(\tau,x)\|\}.\end{aligned} \tag{3.32}$$

Now, assume $\alpha\geq 1$. Then, we can derive a complementary bound

$$\begin{aligned} I_2 &\leq \frac{\sqrt{t^{\alpha-1}+s^{\alpha-1}}}{\Gamma(\alpha)}\left(\int_0^s ((t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1})\cdot\|g(\tau,X_k(\tau))\|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{(t^{\alpha-1}+s^{\alpha-1})((t^\alpha-s^\alpha)-(t-s)^\alpha)}}{\Gamma(\alpha)\sqrt{\alpha}}\sup_{\substack{\tau\in[0,T]\\ x\in B_M(x_0)}}\{\|g(\tau,x)\|\}, \quad \alpha\geq 1.\end{aligned} \tag{3.33}$$

In the case where $\frac{1}{2}<\alpha<1$, observe that $2\alpha>1$. As $u\to 1^-$, the numerator $(1-u^{\alpha-1})^2$ in the integrand of (3.32) vanishes faster than the denominator $(1-u)^{2\alpha}$, yielding

$$\lim_{u\to 1^-}\frac{(1-u^{\alpha-1})^2}{(1-u)^{2\alpha}}=0.$$

Hence, the integrand remains bounded and we obtain

$$I_2 \leq \frac{(t-s)^{2\alpha-1}}{\Gamma(\alpha)}\max_{u\in[0,1)}\left\{\frac{(1-u^{\alpha-1})^2}{(1-u)^{2\alpha}}\right\}\sup_{\substack{\tau\in[0,T]\\ x\in B_M(x_0)}}\|g(\tau,x)\|, \quad \text{for } \alpha\in\left(\frac{1}{2},1\right). \tag{3.34}$$

In light of the estimates in (3.31), (3.33), and (3.34), and incorporating them into (3.30), we establish the equicontinuity of the sequence $\{X_k(t)\}_{k\geq 1}$.

Applying the Arzelà–Ascoli theorem, we deduce the existence of a uniformly convergent subsequence $\{X_{k_n}(t)\}_{n\geq 1}$ on the interval $I$, converging to a continuous function $\overline{X}(t)$. Furthermore, since the subsequence $\{X_{k_n}(t)\}_{n\geq 1}$ satisfies (3.29), its limit satisfies

$$\widehat{X}(t)=\sum_{l=0}^{\lfloor\alpha\rfloor}\frac{X_0^{(l)}}{l!}t^l+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,\widehat{X}(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,\widehat{X}(s))dB(s), \quad t\in I. \tag{3.35}$$

From the previous analysis showing that $|X_{k+1}(t)-X_k(t)|\to 0$ uniformly on $I$, we conclude that $\overline{X}(t)=\widehat{X}(t)$. Given that $\widehat{X}(t)$ is a solution to Eq. (3.25), and this solution is unique, every uniformly convergent subsequence must converge to the same function.

Therefore, the entire sequence $\{X_k(t)\}_{k\geq 1}$ converges uniformly on $I$ to the unique solution of Eq. (3.25), thereby completing the proof. □

### 3.3. Pathwise uniqueness of solutions to nonlinear SDEs

We now turn our attention to the classical nonlinear stochastic differential equation (SDE) given by

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \quad t \in (0, T], \tag{3.36}$$

with initial condition $X(0) = X_0$, where all notations are consistent with those previously defined. As a direct consequence of Theorem 3.9, we derive the following result.

**Corollary 3.10.** *In Theorem 3.9, assume that $0 < \beta < 2^{1-p}T^{1-\frac{p}{2}}$, for $p \geq 2$, and that all other notations and assumptions are as defined in Theorem 3.9. Suppose the continuous functions $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $g : I \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the following conditions:*

$$|f(t,x) - f(t,y)|^p \leq h(t)\,\psi(|x-y|^p), \quad t \in I, \ x, y \in \mathbb{R}^n, \tag{3.37}$$

$$\|g(t,x) - g(t,y)\|^p \leq \beta\,\frac{\eta'(t)}{\eta(t)}\,|x-y|^p, \quad t \in (0,T], \ x, y \in \mathbb{R}^n, \tag{3.38}$$

$$|f(t,x)|^p \vee \|g(t,x)\|^p \leq \phi(|x|^p), \quad (t,x) \in I \times \mathbb{R}^n. \tag{3.39}$$

*Then, for any random variable $X_0 \in L^p(\Omega;\mathbb{R}^n)$ that is independent of the Brownian motion $\{B(t)\}_{t\in I}$, the solution to Eq. (3.36) is pathwise unique.*

We now compare Corollary 3.10 with a related result by Liu & Liu [37]. Define the function class

$$\mathcal{L} := \left\{ u \in C^1((0,t_0],\mathbb{R}^+) \;\middle|\; \lim_{t\to 0} u(t) = 0, \ \liminf_{t\to 0} u'(t) > 0, \ \text{and } u'(t) > 0 \text{ for all } t \in (0,t_0] \right\}.$$

Liu & Liu [37] established a uniqueness result for solutions of the SDE (3.36) under the following conditions.

**Theorem 3.11** (*Theorem 3.4, [37]*). *Assume constants $\beta \in (0, 1/2)$, $K > 0$, and $u \in \mathcal{L}$ such that $\lim_{t\to 0} u'(t) = +\infty$. Let $\omega : [0,t_0]\times[0,\infty) \to [0,\infty)$ be non-decreasing and concave in its second argument and such that $\omega(t,x)/u(t)$ is continuous on $[0,t_0]\times[0,\infty)$. Suppose $x(t) \equiv 0$ is the unique solution to*

$$x'(t) = \alpha\frac{\omega(t,x(t))}{u(t)}, \quad x(0) = 0,$$

*on $[0,t_0]$, where $\alpha = \frac{2t_0 u(t_0)}{1-2\beta}$. Assume further that $\alpha\omega(t,x)/u(t)$ admits an upper function. If $f, g : [0,t_0]\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous and satisfy*

$$\begin{aligned} |f(t,x) - f(t,y)|^2 &\leq \omega(t,|x-y|^2), \quad t \in [0,t_0], \ x,y \in \mathbb{R}^n, \\ |g(t,x) - g(t,y)|^2 &\leq \beta\frac{u'(t)}{u(t)}|x-y|^2, \quad t \in (0,t_0], \ x,y \in \mathbb{R}^n, \\ |f(t,x)|^2 + |g(t,x)|^2 &\leq K^2(1+|x|^2), \quad t \in [0,t_0], \ x \in \mathbb{R}^n, \end{aligned} \tag{3.40}$$

*then pathwise uniqueness holds for any $X_0 \in L^2(\Omega;\mathbb{R}^n)$ independent of the Brownian motion $\{B(t)\}_{t\in[0,t_0]}$.*

We now show that our result in Corollary 3.10 generalizes Theorem 3.11 (for $p = 2$). We verify that assumptions (3.40) imply (3.37)–(3.39):

**(i)** To verify (3.37), define the auxiliary function $R(t,x) := \omega(t,x)/x, x > 0$. Since $\omega(t,\cdot)$ is assumed to be non-decreasing and concave with $\omega(t,0) = 0$, we have

$$\lim_{x\to 0^+} R(t,x) = \lim_{x\to 0^+}\frac{\omega(t,x)}{x} = \omega_x^+(t,0),$$

where $\omega_x^+(t,0)$ denotes the right derivative at $x = 0$, which exists due to concavity. Therefore, $R(t,x)$ is continuous on $(0,x_0]$ for any fixed $x_0 > 0$, and we can define

$$C_1 := \sup_{t\in[0,t_0]}\sup_{x\in(0,x_0]} R(t,x) < \infty.$$

For the case of large $x$, say $x \geq x_0$, concavity again implies that $\omega(t,x)$ grows at most linearly. Specifically, by the supporting line property of concave functions, we have

$$\omega(t,x) \leq \omega(t,x_0) + \omega_x^+(t,x_0)(x-x_0),$$

which gives

$$R(t,x) = \frac{\omega(t,x)}{x} \leq \frac{\omega(t,x_0)}{x} + \omega_x^+(t,x_0)\left(1 - \frac{x_0}{x}\right).$$

Since the right-hand side is bounded on $x \in [x_0, \infty)$, uniformly in $t \in [0, t_0]$, we define

$$C_2 := \sup_{t\in[0,t_0]} \sup_{x\geq x_0} R(t,x) < \infty.$$

Combining both regions, we conclude that

$$\omega(t,x) \leq Cx, \quad \text{for all } (t,x) \in [0,t_0] \times [0,\infty),$$

where $C := \max\{C_1, C_2\}$. Therefore, the condition (3.37) is satisfied with $h(t) := C$ and $\psi(u) := u$, which is linear and trivially satisfies the Osgood condition.

**(ii)** To verify (3.38), From the assumptions $u \in \mathcal{L}$ and $\lim_{t\to 0} u'(t) = +\infty$, we know that $u(t) \to 0$ as $t \to 0$ and increases rapidly. In particular, there exists $c > 0$ and $\delta > 0$ such that

$$u(t) > c\sqrt{t}, \quad \text{for all } t \in (0,\delta).$$

It follows that

$$\int_0^\delta \frac{1}{u(t)}\, dt \leq \frac{2}{c}\sqrt{\delta} < \infty.$$

On the interval $[\delta, T]$, $u(t)$ is continuous and strictly positive, so $1/u(t)$ is bounded and Lebesgue integrable. Thus,

$$\int_0^T \frac{1}{u(t)}\, dt < \infty.$$

Setting $\eta(t) := u(t)$, the coefficient $\frac{\eta'(t)}{\eta(t)}$ in (3.38) matches the assumption in (3.40), and so the inequality (3.38) is satisfied.

**(iii)** To verify (3.39), from the growth condition in Theorem 3.11:

$$|f(t,x)|^2 + |g(t,x)|^2 \leq K^2(1+|x|^2),$$

we deduce in particular that

$$|f(t,x)|^2 \vee \|g(t,x)\|^2 \leq K^2(1+|x|^2).$$

Therefore, we may take $\phi(r) := K^2(1+r)$, which is continuous, non-decreasing, and satisfies the required growth condition in (3.39).

In summary, all the assumptions of Corollary 3.10 are fulfilled under those of Theorem 3.11. Furthermore, Corollary 3.10 extends Theorem 3.11 in several ways: it permits general nonlinear continuity moduli $\psi$, supports more flexible growth conditions through general functions $\phi$, and applies to all $p \geq 2$, broadening its scope beyond the quadratic case.

**Remark 3.12.** If the diffusion term is absent, i.e., $g \equiv 0$, then Eq. (3.36) simplifies to a deterministic ordinary differential equation (ODE). In this setting, the requirement for the Nagumo-like condition can be relaxed by replacing the condition $\lim_{t\to 0^+} \eta'(t) = +\infty$ with $\liminf_{t\to 0^+} \eta'(t) > 0$, together with

$$\lim_{(t,X)\to(0,X_0)} \frac{f(t,X) - f(0,X_0)}{\eta'(t)} = 0.$$

To use this modified condition in order to establish that $\lim_{t\to 0^+} \frac{u_k(t)}{\eta(t)} = 0$, as outlined in (3.13), consider $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$ as two solutions to the given ODE. Let $f = (f_1, f_2, \ldots, f_n)$, and apply l'Hôpital's rule to obtain

$$\lim_{t\to 0^+} \frac{X_i(t) - Y_i(t)}{\eta(t)} = \lim_{t\to 0^+} \frac{f_i(t,X(t)) - f_i(t,Y(t))}{\eta'(t)} = 0, \quad \text{for all } i = 1,2,\ldots,n,$$

which leads to the conclusion that

$$\lim_{t\to 0^+} \frac{u_k(t)}{\eta(t)} = \lim_{t\to 0^+} \left(I_k(t)|X(t)-Y(t)|^{p-1}\right)\left(\frac{|X(t)-Y(t)|}{\eta(t)}\right) = 0.$$

### *3.4. Pathwise uniqueness of solutions to nonlinear FDEs*

Setting $g \equiv 0$ and $\alpha \in (0,2]$ in Eq. (1.1), we study the deterministic fractional differential equation driven purely by the Caputo derivative and drift function $f$:

$$^C D^\alpha x(t) = f(t,x(t)), \quad x(0) = x_0, \;\; x'(0) = x_1, \quad t \in (0,T]. \tag{3.41}$$

**Theorem 3.13.** *Let $T > 0$. Suppose $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies:*

*1.*

$$|f(t,x) - f(t,y)| \leq h(t)\psi\big(|x-y|\big), \quad h \in L^1(0,T), \quad t \in [0,T], \;\; x,y \in \mathbb{R}^n \quad \textit{(Osgood-type continuity)}$$

2.

$$|f(t,x)| \le \phi\big(|x|\big), \quad t \in [0,T],\ x \in \mathbb{R}^n \quad \textit{(General growth)}$$

*Here* $\phi, \psi : [0,\infty) \to [0,\infty)$ *are continuous, non-decreasing. Moreover,* $\psi(0) = 0$, $\psi(u) > 0$ *for* $u > 0$, *and satisfies the divergence condition*

$$\int_{0^+} \frac{du}{\psi(u)} = +\infty.$$

*Then the fractional IVP (3.41) admits at least one continuous solution on* $[0,T]$, *which is moreover pathwise unique.*

**Proof.** Existence follows by Theorem 3.9. Uniqueness is shown as follows.

Let $x(t)$ and $y(t)$ be two solutions with identical initial data. Define

$$u(t) := |x(t) - y(t)|.$$

Subtract the integral forms and apply the triangle inequality

$$u(t) \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, \psi\big(u(s)\big)\, ds.$$

Defining the kernel function

$$K(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

which is continuous and finite on $[0,T]$, we obtain

$$u(t) \le K(t)\psi\big(u(t)\big), \quad t \in [0,T].$$

If $u \not\equiv 0$, let $t_1 > 0$ be the first time $u(t_1) > 0$. Then a standard Osgood argument shows

$$\int_0^{u(t_1)} \frac{dr}{\psi(r)} \le \int_0^{t_1} K(s)\, ds < \infty,$$

contradicting the divergence of $\int_{0^+}(1/\psi)$. Hence $u \equiv 0$ and uniqueness holds. □

**Remark 3.14.** In the integer-order case $\alpha = 2$, Chu [36] studied the second-order ODE

$$u''(t) = \frac{F(u(t))}{\cosh^2(t)} - \frac{2\omega \sinh(t)}{\cosh^3(t)}, \quad t > t_0,$$

where $F$ satisfies the Osgood-type condition (see (1.2)) and a growth condition of the form $|F(x)| \le g(|x|)$, with $g$ continuous and non-decreasing. Chu's result established existence and uniqueness under these structural and asymptotic assumptions, in a setting motivated by fluid dynamics, particularly modeling Arctic gyres.

In contrast, Theorem 3.13–a deterministic, drift-only instance of our general stochastic Theorem 3.1–significantly broadens the framework. Our result applies to arbitrary fractional orders $\alpha \in (0,2]$ (and can, in fact, be formulated for every $\alpha > 0$, as discussed in Remark 3.2); it also permits time-dependent (non-autonomous) drift functions $f(t,x)$, and allows general growth conditions through non-decreasing functions $\phi(|x|)$, rather than relying on specific decay structures such as $\cosh^{-2}(t)$.

Hence, our uniqueness theorem not only recovers Chu's result in the special case $\alpha = 2$ with drift $f(t,x)$ as in Chu's model, but also substantially extends it to fractional-order dynamics under broader regularity and growth assumptions.

## 4. Some concrete examples

In this section, we provide two illustrative examples to demonstrate the applicability of Theorem 3.1 and Theorem 3.9 to certain classes of fractional stochastic differential equations (FSDEs). In particular, the second example highlights a case where the drift and diffusion coefficients do not satisfy the standard Lipschitz condition.

**Example 4.1.** We consider the following McKean–Vlasov-type fractional stochastic differential equation:

$$^{C}D_0^{\alpha} X(t) = -a(X(t) - \mathbb{E}[X(t)]) + \sigma \frac{dB_t}{dt}, \quad X(0) = 1, \quad t \in (0,T], \tag{4.1}$$

where $\alpha \in (0,1)$, $\mathbb{E}[X(t)]$ is the expected value of the process at time $t$ (representing the mean-field interaction), $a > 0$ is a drift constant, $\sigma > 0$ is the diffusion coefficient, and $B_t$ is a standard Brownian motion.

To approximate the solution numerically, we employ a simplified Euler-type scheme [47] adapted for fractional dynamics:

$$X_{k+1}^{(j)} = X_k^{(j)} + \frac{\Delta t^{\alpha}}{\Gamma(1+\alpha)} \left[-a(X_k^{(j)} - \overline{X}_k)\right] + \sigma \Delta B_k^{(j)}, \tag{4.2}$$

where $\overline{X}_k = \frac{1}{M}\sum_{j=1}^{M} X_k^{(j)}$ is the empirical mean at time step $k$, $M$ is the number of simulated paths, and $\Delta B_k^{(j)}$ are increments of Brownian motion.

This numerical scheme captures the combined influence of memory effects (due to the fractional derivative) and mean-field interactions. We simulate $M = 500$ sample paths and compute the average trajectory of $X(t)$ over the time interval $[0, 100]$, using specified values for $a$ and $\sigma$, and with a fractional order of $\alpha = 0.9$ (see Fig. 1).

To verify the applicability of Theorem 3.1 to Eq. (4.1), it suffices to confirm the validity of inequalities (3.1)–(3.3). Let us take

$$f(t,x,\mu) = -(x-\mu), \qquad g(t,x,\mu) = 0.5, \quad x, y \in \mathbb{R}, \ \mu, \nu \in \mathcal{M}_p(\mathbb{R}), \ 2.5 < p < 5.$$

We then observe

$$|f(t,x,\mu) - f(t,y,\nu)|^p \le 2^{p-1}(|x-y|^p + |\mu - \nu|^p) \le 2^{p-1}\left(|x-y|^p + \mathbb{W}_p^p(\mu,\nu)\right),$$
$$\|g(t,x,\mu) - g(t,y,\nu)\| = 0.$$

Note that the condition $2.5 < p < 5$ ensures the inequality $\alpha > \max\left\{\frac{1}{2} + \frac{1}{p}, 1 - \frac{1}{2p}\right\}$, where $\alpha = 0.9$.

Since $\mu$ and $\nu$ denote the means of the probability measures $\mu$ and $\nu$, respectively, we employ the following inequality from optimal transport theory:

$$|\mu - \nu| \le \mathbb{W}_p(\mu,\nu), \quad \text{for all } \mu, \nu \in \mathcal{M}_p(\mathbb{R}).$$

Now assume that $\mu \in \mathcal{M}_p(\mathbb{R})$ is a probability measure with a finite $p$-moment. That is, there exists a constant $A > 0$ such that

$$\int_{\mathbb{R}} |y|^p \, \mu(dy) \le A,$$

and we choose $A$ large enough so that $A > 2^{1-2p}$. Then, using the inequality

$$|x-y|^p \le 2^{p-1}(|x|^p + |y|^p),$$

we obtain

$$|x-\mu|^p := \int_{\mathbb{R}} |x-y|^p \, \mu(dy) \le 2^{p-1}(|x|^p + A).$$

This further implies

$$|f(t,x,\mu)|^p \le 2^{p-1}(|x|^p + A),$$

and consequently,

$$\max\{|x-\mu|^p, (0.5)^p\} \le 2^{p-1}(|x|^p + A).$$

Therefore, we obtain the following bound

$$|f(t,x,\mu)|^p \vee \|g(t,x,\mu)\|^p \le \max\{|x-\mu|^p, (0.5)^p\} \le \phi(|x|^p),$$

where $\phi(x) := 2^{p-1}(1+A)(1+x)$.

This verifies inequality (3.3). The other assumptions of Theorem 3.1 can be verified in a similar manner and are readily satisfied in this setting.

Hence, Eq. (4.1) admits a unique solution in the $L^p$-sense for $2.5 < p < 5$, obtained via the successive approximation scheme described in (4.2) (see Fig. 1). The resulting dynamics, as shown in Fig. 1, exhibit stable behavior and confirm the mean-field damping effect characteristic of the McKean–Vlasov framework.

**Example 4.2.** Consider the following two-dimensional fractional stochastic differential equation defined on the interval $I = [0, \pi]$:

$$^{C}D_0^{1.1} x(t) = f(t, x(t)) + g(t, x(t))\frac{dB(t)}{dt}, \quad t \in (0, \pi], \tag{4.3}$$

where $x(0) = (x_1(0), x_2(0)) \in \mathbb{R}^2$ is a given random variable.

The components of the drift term $f(t,(x_1,x_2)) = \big(f_1(t,x_1), f_2(t,x_2)\big)$ and the diffusion term $g(t,(x_1,x_2)) = \big(g_1(t,x_1), g_2(t,x_2)\big)$ are defined by

$$f_i(t,x_i) = \begin{cases} 0, & t \in I, \ x_i \le 0, \\ x_i(-p\ln x_i)^q \sin^i t, & t \in I, \ x_i \in (0, e^{-1/p}), \\ e^{-1/p}\sin^i t, & t \in I, \ x_i \ge e^{-1/p}, \end{cases} \tag{4.4}$$

$$g_i(t,x_i) = \begin{cases} 0, & t \in I, \ x_i \le 0, \\ \dfrac{1}{\lambda} x_i t^{-1/p}, & t \in I \setminus \{0\}, \ x_i \in (0, t), \\ \dfrac{1}{\lambda} t^{(p-1)/p}, & t \in I, \ x_i \ge t, \end{cases} \tag{4.5}$$

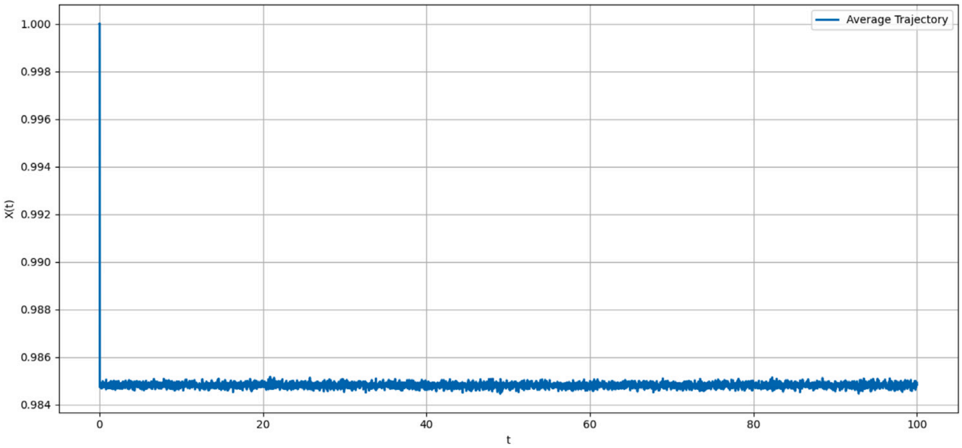


**Fig. 1.** Average trajectory of Eq. (4.1) over the time interval $[0, 100]$, with initial value $x_0 = 1$, time step $\Delta t = 0.02$, and parameters $\alpha = 0.9$, $a = 1$, and $\sigma = 0.5$. The simulation was performed over $N = 5000$ time steps and $M = 500$ sample paths. The stability of the average path illustrates the well-posedness of the system under Lipschitz conditions.

for $i = 1, 2$, where $2 \le p < q^{-1}$, and

$$\lambda > 2^{1-1/p}\pi^{0.6-1/p}\Gamma(1.1)^{-1}\left(\frac{p-2}{1.2p-2}\right)^{0.5-1/p}.$$

For instance, if $p = 2$, then the condition simplifies to $\lambda > \sqrt{2}\pi^{0.1}\Gamma(1.1)^{-1} \approx 1.667$; and if $p = 3$, then $\lambda > 2.094$.

We emphasize that the stated lower bound for the positive constant $\lambda$ is essential to ensure the validity of a crucial parameter that will be specified later in this example.

Now, we claim that the functions $f$ and $g$, defined above, satisfy the assumptions of Theorem 3.9.

To verify condition (3.26), let $t \in I$ and suppose $x_i > y_i$. The difference between the values of $f_i$ at $(t, x_i)$ and $(t, y_i)$ is given by

$$f_i(t,x_i) - f_i(t,y_i) = \begin{cases} 0, & x_i \le 0, \\ x_i(-p\ln x_i)^q \sin^i t, & y_i \le 0 < x_i < e^{-1/p}, \\ e^{-1/p}\sin^i t, & y_i \le 0,\ \ x_i \ge e^{-1/p}, \\ (x_i(-p\ln x_i)^q - y_i(-p\ln y_i)^q)\sin^i t, & 0 < y_i < x_i < e^{-1/p}, \\ (e^{-1/p} - y_i(-p\ln y_i)^q)\sin^i t, & 0 < y_i < e^{-1/p} \le x_i, \\ 0, & e^{-1/p} \le y_i \le x_i. \end{cases} \tag{4.6}$$

On the other hand, for $t \in I$, $x_i \in (0, e^{-1/p})$, we compute

$$\begin{cases} \dfrac{\partial f_i}{\partial x_i}(t,x_i) = -p\sin^i t \cdot \dfrac{\ln x_i + q}{(-p\ln x_i)^{1-q}}, \\ \dfrac{\partial^2 f_i}{\partial x_i^2}(t,x_i) = p^2 q \sin^i t \cdot \dfrac{\ln x_i - (1-q)}{(-p\ln x_i)^{2-q}}, \end{cases} \tag{4.7}$$

which shows that $f_i$ is strictly increasing and concave over $(0, e^{-1/p})$, since $\partial^2 f_i/\partial x_i^2 < 0$ on this interval.

Hence, for $0 < y_i < x_i \le e^{-1/p}$, we obtain

$$\begin{aligned} f_i(t,x_i) - f_i(t,y_i) &= \left(x_i(-p\ln x_i)^q - y_i(-p\ln y_i)^q\right)\sin^i t \\ &= \int_{y_i}^{x_i} \frac{\partial f_i}{\partial x_i}(t,v)\,dv = \int_0^{x_i-y_i} \frac{\partial f_i}{\partial x_i}(t,v+y_i)\,dv \\ &\le \int_0^{x_i-y_i} \frac{\partial f_i}{\partial x_i}(t,v)\,dv = f_i(t,x_i-y_i), \quad i = 1,2. \end{aligned} \tag{4.8}$$

Now, combining (4.6) and (4.8), let $x = (x_1, x_2),\ y = (y_1, y_2) \in [0, e^{-1/p}]^2$ and set $r_i = x_i - y_i > 0$, $i = 1, 2$. Then, by Jensen's and generalized Minkowski inequalities:

$$\begin{aligned} |f(t,x) - f(t,y)|^p &= \left(|f_1(t,x_1) - f_1(t,y_1)|^2 + |f_2(t,x_2) - f_2(t,y_2)|^2\right)^{p/2} \\ &\le \left(f_1^2(t,r_1) + f_2^2(t,r_2)\right)^{p/2} \\ &= \left(r_1^2\sin^2 t(-p\ln r_1)^{2q} + r_2^2\sin^4 t(-p\ln r_2)^{2q}\right)^{p/2} \\ &\le 2^{\frac{p}{2}-1}|\sin t|^p\left(r_1^p(-p\ln r_1)^{pq} + r_2^p(-p\ln r_2)^{pq}\right), \quad 0 < r_i \le e^{-1/p},\ i = 1,2. \end{aligned} \tag{4.9}$$

Define $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(r) = \begin{cases} 0, & r = 0, \\ r(-\ln r)^{pq}, & 0 < r < e^{-1}, \\ \frac{1}{2}(r + e^{-1}), & r \geq e^{-1}, \end{cases}$$

which is continuous, nondecreasing, and concave since $\psi''(r) \leq 0 < \psi'(r)$ for $r > 0$. Furthermore, for any $\delta > 0$, and $\epsilon \in (0,\delta)$,

$$\int_{0^+} \frac{dr}{\psi(r)} = \lim_{\epsilon\to 0^+} \int_\epsilon^\delta \frac{dr}{r(-\ln r)^{pq}} \overset{u=\ln r}{=} \lim_{\epsilon\to 0^+} \int_{\ln\epsilon}^{\ln\delta} (-u)^{-pq} du = \frac{1}{1-pq}\left[(-\ln\epsilon)^{1-pq} - (-\ln\delta)^{1-pq}\right] \to +\infty.$$

Also, since $\psi$ is nondecreasing,

$$\max\{\psi(r_1^p), \psi(r_2^p)\} \leq \psi\left((r_1^2 + r_2^2)^{p/2}\right).$$

Thus, continuing from (4.9) we derive

$$\begin{aligned} |f(t,x) - f(t,y)|^p &\leq 2^{\frac{p}{2}} |\sin t|^p \cdot \frac{\psi(r_1^p) + \psi(r_2^p)}{2} \\ &\leq 2^{\frac{p}{2}} |\sin t|^p \cdot \psi\left((r_1^2 + r_2^2)^{p/2}\right) = h(t)\psi\left(|x-y|^p\right), \end{aligned}$$

where $h(t) := 2^{p/2}|\sin t|^p$. Hence, condition (3.26) holds:

$$|f(t,x) - f(t,y)|^p \leq h(t)\,\psi(|x-y|^p).$$

Although $f$ is not Lipschitz, it satisfies a generalized growth condition. In fact, using (4.7), we observe that

$$\begin{aligned} \lim_{x_i\to 0^+} \frac{\partial f_i}{\partial x_i}(t,x_i) &= -p\sin^i t \lim_{x\to 0^+} \frac{\ln x + q}{(-p\ln x)^{1-q}} \\ &\overset{\text{L'Hôpital}}{=} \frac{\sin^i t}{1-q} \lim_{x\to 0^+} (p\ln\tfrac{1}{x})^q = +\infty. \end{aligned}$$

On the other hand, to verify condition (3.27), consider $t \in I$ and assume $x_i > y_i$ for $i = 1,2$. Then,

$$g_i(t,x_i) - g_i(t,y_i) = \begin{cases} 0, & x_i \leq 0, \\ \frac{1}{\lambda} x_i t^{-1/p}, & y_i \leq 0 < x_i < t, \\ \frac{1}{\lambda} t^{(p-1)/p}, & y_i \leq 0,\ t \leq x_i, \\ \frac{1}{\lambda} t^{-1/p}(x_i - y_i), & 0 < y_i < x_i < t, \\ \frac{1}{\lambda} t^{-1/p}(t - y_i), & 0 < y_i < t \leq x_i, \\ 0, & t \leq y_i \leq x_i. \end{cases}$$

Consequently, we obtain the following estimate:

$$\|g(t,x) - g(t,y)\|^p = \left(|g_1(t,x_1) - g_1(t,y_1)|^2 + |g_2(t,x_2) - g_2(t,y_2)|^2\right)^{p/2} \leq \frac{1}{\lambda^p t}|x-y|^p, \quad \forall t \in (0,\pi].$$

Define an absolutely continuous function $\eta(t) = t^\gamma$, with $\gamma \in [\beta^{-1}\lambda^{-p}, 1)$, where $\beta$ is any fixed constant satisfying

$$\lambda^{-p} < \beta < 2^{1-p}\pi^{1-0.6p}\Gamma(1.1)^p \left(\frac{p-2}{1.2p-2}\right)^{1-\frac{p}{2}}.$$

This choice guarantees that $\eta$ satisfies both

$$\eta(t) \leq \beta\lambda^p t\eta'(t), \quad \text{for all } t \in (0,\pi], \quad \text{and} \quad \lim_{t\to 0^+} \eta'(t) = +\infty.$$

Hence, $\eta$ fulfills the required properties in Theorem 3.9, and the function $g$ satisfies condition (3.27), despite not being Lipschitz continuous.

Indeed, for small $t > 0$, the point $(t,t) \in \mathbb{R}^2$ lies in the region where each $x_i \in (0,t)$. Then, from the definition of $g_i$,

$$g(t,(t,t)) = \left(\frac{1}{\lambda}t^{1-\frac{1}{p}}, \frac{1}{\lambda}t^{1-\frac{1}{p}}\right), \quad g(t,(0,0)) = (0,0).$$

Thus, we have

$$\|g(t,(t,t)) - g(t,(0,0))\| = \frac{\sqrt{2}}{\lambda}t^{1-\frac{1}{p}}, \quad \|(t,t) - (0,0)\| = \sqrt{2}t,$$

and hence,

$$\lim_{t\to 0^+} \frac{\|g(t,(t,t)) - g(t,(0,0))\|}{\|(t,t) - (0,0)\|} = \lim_{t\to 0^+} \frac{1}{\lambda}t^{-1/p} = +\infty,$$

confirming the non-Lipschitz nature of $g$.

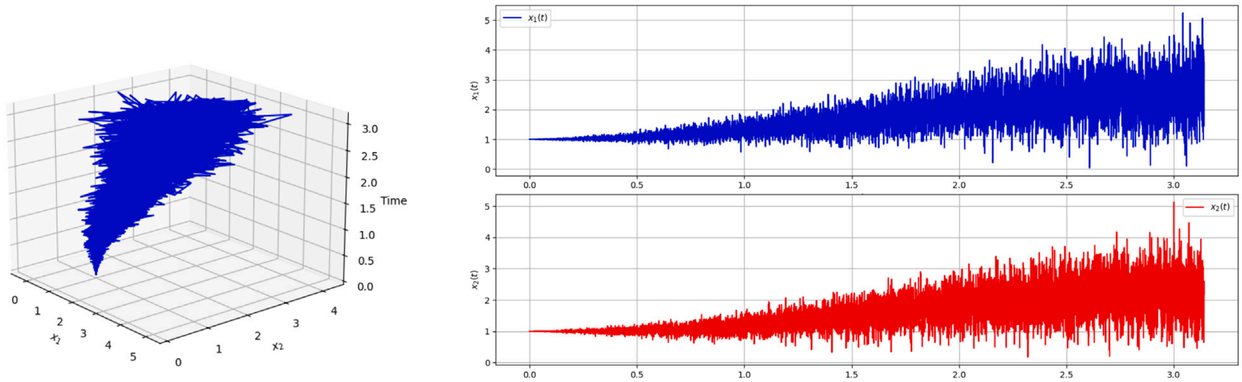


**Fig. 2.** Component-wise trajectories $x_1(t)$ (top right) and $x_2(t)$ (bottom right), along with a 3D trajectory plot (left) of the solution to Eq. (4.3), with initial condition $x_0 = (1, 1)$, step size $dt = 0.0001\pi$, and parameters $p = 3$, $q = \frac{1}{4}$, and $\lambda = 3$. These simulations illustrate the evolution of the solution under drift and diffusion terms that do not satisfy the classical Lipschitz condition. Despite the time-singular behavior and nonlinearity, the solution remains bounded and follows a unique trajectory in $L^p$, demonstrating the well-posedness and uniqueness guaranteed by Theorem 3.9.

Furthermore, we estimate the growth of $f$ and $g$ as follows:

$$|f(t,x)|^p \vee \|g(t,x)\|^p \le 2^{p/2} \max\left\{e^{-1}, \frac{\pi^{p-1}}{\lambda^p}\right\}.$$

Therefore, condition (3.28) is also satisfied.

In conclusion, all the assumptions of Theorem 3.9 are fulfilled. Although neither $f$ nor $g$ satisfies the classical Lipschitz condition, they fall within the broader class of functions accommodated by the theorem. Hence, Eq. (4.3) admits a unique solution in the $L^p$-sense, constructed through the successive approximation method described in (3.29) (see Fig. 2).

It is important to note that, although the sample paths displayed in Fig. 2 appear irregular or highly variable, this behavior should not be interpreted as instability in a theoretical sense—such as non-uniqueness or finite-time blow-up. Instead, these visual characteristics reflect the inherent complexity induced by the nonlinear, non-Lipschitz structure of the drift and diffusion terms, as well as the time-singular behavior near $t = 0$. The system remains well-posed within our framework and provides a strong demonstration of the robustness of our uniqueness result under relaxed assumptions.

**Remark 4.3.** In Example 4.2, although the coefficients $f$ and $g$ are not Lipschitz and exhibit singular behavior near $t = 0$, this does not violate the assumptions of Theorem 3.9. In particular, the drift term contains derivatives that blow up near $x_i = 0$, and the diffusion term includes $t^{-1/p}$, which diverges as $t \to 0^+$. Nevertheless, the associated growth is effectively controlled by the carefully chosen weight functions: $h(t) = 2^{p/2}|\sin t|^p$ (which is integrable on $(0, \pi]$) and $\eta(t) = t^\gamma$, with $\gamma \in (0, 1)$, whose derivative diverges as $t \to 0^+$. These functions absorb the singularities and ensure the conditions (3.26) and (3.27) are satisfied uniformly. This confirms that the example lies within the scope of the theorem, even though classical Lipschitz conditions are not met.

## 5. Conclusion and future directions

In this paper, we investigated pathwise uniqueness for a class of McKean–Vlasov stochastic differential equations (SDEs) driven by Brownian motion, formulated in the $L_p$-setting. By combining Osgood- and Nagumo-type growth conditions, we relaxed the classical Lipschitz assumptions on both drift and diffusion terms. This enabled us to establish uniqueness results under non-Lipschitz, possibly sublinear, and distribution-dependent settings—broadening the scope of well-posedness theory for fractional SDEs.

Looking ahead, several promising directions for future research emerge. First, a natural extension involves SDEs driven by Lévy processes or other non-Gaussian noise sources. In such settings, higher-order moment estimates are crucial, and the $L_p$-framework developed here, with $p \ge 2$, offers a robust foundation. The truncation and auxiliary-function techniques presented in this work could, in principle, be adapted to accommodate jump-type dynamics, possibly in combination with suitable nonlocal or integro-differential inequalities.

Second, an important direction is to explore systems exhibiting long-memory effects more explicitly. These include models driven by fractional Brownian motion or incorporating distributed-order derivatives, which appear in physics, biology, and finance. While such systems are generally non-Markovian, the structural similarity to the Caputo framework used here suggests that our techniques may be extended, with appropriate modifications, to this broader class.

Third, although our main result assumes a lower bound on the fractional order $\alpha > \max\left\{\frac{1}{2} + \frac{1}{p},\ 1 - \frac{1}{2p}\right\}$, extending the theory to smaller values of $\alpha$ (e.g., $\alpha = \frac{1}{2}$) remains an open challenge. This would be particularly relevant in singular regimes where current integrability arguments break down. Further refinement of the auxiliary-function framework may help overcome these technical obstacles.

Finally, our setting naturally raises questions about the possible emergence of stochastic bifurcations or chaotic dynamics in non-Lipschitz, memory-sensitive systems. While this lies beyond the present scope, it presents an intriguing avenue for future theoretical investigation.

We hope that this work contributes meaningfully to the theory of fractional McKean–Vlasov SDEs and that the methodological tools developed here serve as a flexible starting point for further research in more general stochastic systems.

## CRediT authorship contribution statement

**Ehsan Pourhadi:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Software, Investigation, Formal analysis. **Chenkuan Li:** Writing – review & editing, Validation, Methodology, Investigation, Formal analysis.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Data availability

No data was used for the research described in the article.

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