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A FIXED-POINT APPROACH TO THE EXISTENCE AND UNIQUENESS OF A FRACTIONAL NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS AND FUNCTIONAL BOUNDARY CONDITION

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We study the existence and uniqueness of solutions for a new fractional nonlinear integro-differential equation with variable coefficients and a functional boundary condition using Krasnoselskii's fixed point theorem and Banach's contractive principle. The approach relies on an inverse operator in a Banach space, the Mittag-Leffler function and an implicit integral equation. The technique used has many applications to studying various nonlinear integral or differential equations, including partial differential equations, with initial or boundary conditions. Several illustrative examples are provided to show applications of our main theorems by computing approximate values of the Mittag-Leffler functions.

1. Introduction

The Riemann–Liouville fractional integral I^β of order $\beta \in \mathbb{R}^+$ is defined for the function $y(x)$ [8; 20] as

$$(I^\beta y)(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-\tau)^{\beta-1} y(\tau) d\tau.$$

In particular from [9],

$$I^0 y = y.$$

The Liouville–Caputo fractional derivative ${}_C D^\gamma$ of order $\gamma \in (1, 2]$ of the function $y(x)$ is defined as [8]

$$({}_C D^\gamma y)(x) = \left(I^{2-\gamma} \frac{d^2}{dx^2} y \right)(x) = \frac{1}{\Gamma(2-\gamma)} \int_0^x (x-\tau)^{1-\gamma} y''(\tau) d\tau.$$

We define the Banach space $C[0, 1]$ of all continuous mappings from $[0, 1]$ into $\mathbb{R}$ with the norm

$$\|y\| = \max_{x \in [0,1]} |y(x)| < +\infty.$$

Let $\alpha \in C[0, 1]$ and $\chi : C[0, 1] \to \mathbb{R}$ be a functional. We will study existence and uniqueness for the following fractional nonlinear integro-differential equation with variable coefficients and a functional boundary condition in $C[0, 1]$:

$$\begin{cases} {}_C D^\gamma y(x) + \alpha(x) I^\beta y(x) = f_1(x, y(x)) + f_2(x, y(x)), & x \in [0, 1], \\ y(0) = 0,\ y(1) = \chi(y), \end{cases} \tag{1-1}$$

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where f_1 and f_2 are continuous functions from $[0, 1] \times \mathbb{R}$ into $\mathbb{R}$, satisfying certain conditions to be given later.

Equation (1-1) with two nonlinear terms on the right-hand side and a variable coefficient on the left, is new and, to the best of our knowledge, has not been previously investigated. Another motivation of considering this equation is to demonstrate how the use of an inverse operator of a bounded integral in a complete space can be used to study the nonlinear fractional integro-differential equation with a functional (which is a mapping from a function space into $\mathbb{R}$) boundary condition.

On the other hand, due to the two nonlinear terms, functional boundary condition and variable coefficient, it seems difficult to present a nicely structured theorem for a stability at the moment, but it is worth considering in the near future.

Here is an example of equation (1-1):

$$\begin{cases} {}_C D^{1.5} y(x) + \frac{x}{1+x^3} I^{0.9} y(x) = \cos(x y^2(x)) + \frac{1}{10(1+x^2)} |\sin x + y(x)|, \quad x \in [0, 1], \\ y(0) = 0,\ y(1) = \theta \int_0^{1/2} |y(x)| dx, \end{cases}$$

where θ is a constant.

The two-parameter Mittag-Leffler function [6; 13] is defined by

$$E_{\beta_1, \beta_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_1 k + \beta_2)},$$

where $z \in \mathbb{C}$, $\beta_1, \beta_2 > 0$.

Fractional nonlinear differential equations with various initial or boundary conditions have been extensively studied due to their strong demands in many scientific fields [19] such as control theory, physics [7], mechanics, chemistry [16] and engineering. Momani and Odibat [18] implemented relatively new analytical techniques, the variational iteration method and the Adomian decomposition method, for solving linear fractional partial differential equations arising in fluid mechanics. Dehghan and Shakeri [4] presented the solution of ordinary differential equations with multipoint boundary value conditions by means of a seminumerical approach which is based on the homotopy analysis method. Singh et al [22] constituted a homotopy algorithm (basically extension of homotopy analysis method with Laplace transform), namely q-homotopy analysis transform method to solve time- and space-fractional coupled Burgers' equations. In the recent years, there has been a significant development in studying uniqueness [10] and existence [2; 23] of fractional differential equations, including PDEs [11; 21]. Li et al. [12] considered a new nonlinear fractional integro-differential equation with arbitrary order and integral boundary condition. Chávez–Vázquez et al [3] developed and presented a fractional integral sliding-mode control scheme based on the Caputo–Fabrizio derivative and the Atangana–Baleanu integral of the Stanford robot for trajectory tracking tasks. Using the Caputo–Fabrizio fractional derivative, Mohammadi et al. [17] presented a numerical simulation for the transmission of disease with respect to the transmission rate and the basic reproduction number in several cases. Beaudin and Li [1] investigated the uniqueness of solutions to a new fractional partial integro-differential equation with a boundary condition by using a recently established matrix Mittag-Leffler function. El-Gendy [5] studied a problem of Hilfer fractional order of an Itô stochastic differential equation with two nonlocal conditions. Recently, Li et al. [14] studied the uniqueness, existence and Hyers–Ulam stability for the following nonlinear fractional integro-differential equation with a functional boundary condition for $1 < \alpha \leq 2$ and constants λ_i using Banach's contractive principle and Leray–Schauder's fixed-point theorem:

$$\begin{cases} {}_C D^{\alpha}\left[\zeta(x) - \sum_{i=1}^{m} \lambda_i I^{\beta_i} \zeta(x) - \sum_{i=1}^{l} I^{\alpha_i} \phi_i(x, \zeta(x))\right] = \psi(x, \zeta(x)), & x \in [0, 1], \\ \zeta(0) = \zeta_1 \in \mathbb{R}, \zeta(1) = \gamma(\zeta), & m, l \in \mathbb{N}, \end{cases}$$

where $\beta_i > 0$, $\alpha_i > 0$, $\phi_i, \psi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are mappings for $i = 1, 2, \dots, l$, and $\gamma : C[0, 1] \to \mathbb{R}$ is a functional.

The following theorem will be used to prove the existence of solutions to equation (1-1).

Theorem 1 (Krasnoselskii's fixed point theorem [15]). *Let X be a Banach space and $X_1 \subset X$ be a closed, convex, and nonempty set. Also, let $\mathcal{N}_1$, $\mathcal{N}_2$ be mappings such that*:

(i) *$\mathcal{N}_1 v + \mathcal{N}_2 z \in X_1$ whenever $v, z \in X1$.*

(ii) *Operator $\mathcal{N}_1$ is continuous and compact.*

(iii) *Mapping $\mathcal{N}_2$ is a contraction.*

Then there exists a $w \in X_1$ satisfying $w = \mathcal{N}_1 w + \mathcal{N}_2 w$.

The remainder of this paper is organized as follows: In Section 2, we will derive an equivalent integral equation of equation (1-1) by an inverse operator of in $C[0, 1]$, and then obtain sufficient conditions for the existence of solutions by Krasnoselskii's fixed point theorem and the Mittag-Leffler function, with two examples making use of our key theorem. In Section 3, we use Banach's contractive principle and the equivalent integral equation to establish the conditions for the uniqueness, with an applicable example.

2. Existence

Theorem 2. *Let $\alpha \in C[0, 1]$, $1 < \gamma \leq 2$, $\beta \geq 0$, f_1 be a continuous and bounded mapping from $[0, 1] \times \mathbb{R}$ to $\mathbb{R}$ and f_2 be a Lipschitz function satisfying the following inequality for a nonnegative constant $\mathscr{L}$*:

$$|f_2(x, y_1) - f_2(x, y_2)| \leq \mathscr{L} |y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}.$$

Furthermore, we assume $\chi : C[0, 1] \to \mathbb{R}$ is a functional satisfying the inequality below for a nonnegative constant $\mathscr{L}_1$

$$|\chi(y_1) - \chi(y_2)| \leq \mathscr{L}_1 \|y_1 - y_2\|,$$

and

$$\mathscr{Q} = \mathscr{L} E_{\beta+\gamma,\gamma+1}(\|\alpha\|) + \frac{\mathscr{L}}{\Gamma(\gamma+1)} E_{\beta+\gamma,1}(\|\alpha\|) < 1.$$

Then equation (1-1) *has at least one solution in $C[0, 1]$.*

Proof. Applying the integral operator I^{β} to both sides of equation (1-1), we get

$$y(x) - y'(0)x + I^{\gamma} \alpha(x) I^{\beta} y(x) = I^{\gamma} f_1(x, y(x)) + I^{\gamma} f_2(x, y(x)),$$

since $y(0) = 0$. From setting $x = 1$, we infer that

$$y'(0) = \chi(y) + I^{\gamma}_{x=1} \alpha(x) I^{\beta} y(x) - I^{\gamma}_{x=1} f_1(x, y(x)) - I^{\gamma}_{x=1} f_2(x, y(x)).$$

So,

$$(1+I^{\gamma}\alpha(x)I^{\beta})y(x)$$
$$= I^{\gamma} f_1(x, y(x)) + I^{\gamma} f_2(x, y(x)) + \chi(y)x + xI^{\gamma}_{x=1}\alpha(x)I^{\beta}y(x) - xI^{\gamma}_{x=1} f_1(x, y(x)) - xI^{\gamma}_{x=1} f_2(x, y(x)).$$

To use the inverse operator method, we first define the operator

$$O = \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k$$

in $C[0, 1]$. Then for any function $\phi \in C[0, 1]$, we have

$$\|O\phi\| = \left\| \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k \phi \right\| \le \|\phi\| \sum_{k=0}^{\infty} \|\alpha\|^k \|I^{k(\beta+\gamma)}\|$$
$$\le \|\phi\| \sum_{k=0}^{\infty} \|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+1)} = \|\phi\| E_{\beta+\gamma,1}(\|\alpha\|) < +\infty.$$

Hence, O is a continuous mapping from $C[0, 1]$ to itself and the series is uniformly convergent. We further show that O is a unique inverse operator of $1 + I^{\gamma}\alpha(x)I^{\beta}$. Indeed,

$$\begin{aligned} O(1+I^{\gamma}\alpha(x)I^{\beta}) &= (1+I^{\gamma}\alpha(x)I^{\beta})O \\ &= \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k + \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^{k+1} \\ &= 1 + \sum_{k=1}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k + \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^{k+1} \\ &= 1 + \sum_{k=0}^{\infty}(-1)^{k+1} (I^{\gamma}\alpha(x)I^{\beta})^{k+1} + \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^{k+1} = 1 \quad \text{(identity operator)}. \end{aligned}$$

Assuming O_1 is another operator satisfying

$$O_1(1+I^{\gamma}\alpha(x)I^{\beta}) = (1+I^{\gamma}\alpha(x)I^{\beta})O_1 = 1,$$

then we derive $O = O_1$ by applying O to both sides. This implies that

$$\begin{aligned} \text{(2-1)} \quad y(x) = &\sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k I^{\gamma} f_1(x, y(x)) + \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k I^{\gamma} f_2(x, y(x)) \\ &+ \chi(y) \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k x + I^{\gamma}_{x=1}\alpha(x)I^{\beta}y(x) \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k x \\ &- I^{\gamma}_{x=1} f_1(x, y(x)) \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k x - I^{\gamma}_{x=1} f_2(x, y(x)) \sum_{k=0}^{\infty}(-1)^k (I^{\gamma}\alpha(x)I^{\beta})^k x \\ = &\mathcal{N}_1 y(x) + \mathcal{N}_2 y(x), \end{aligned}$$

where

$$\mathcal{N}_1 y(x) = \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma f_1(x, y(x)) + \chi(y) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x$$
$$+ I^\gamma_{x=1}\alpha(x) I^\beta y(x) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x - I^\gamma_{x=1} f_1(x, y(x)) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x,$$

and

$$\mathcal{N}_2 y(x) = \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma f_2(x, y(x)) - I^\gamma_{x=1} f_2(x, y(x)) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x.$$

To see $\mathcal{N}_2$ is a contraction, we consider the difference for $y_1, y_2 \in C[0, 1]$,

$$\mathcal{N}_2 y_1(x) - \mathcal{N}_2 y_2(x) = \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma (f_2(x, y_1(x)) - f_2(x, y_2(x)))$$
$$- I^\gamma_{x=1}(f_2(x, y_1(x)) - f_2(x, y_2(x))) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x.$$

Since f_2 is a Lipschitz function satisfying

$$|f_2(x, y_1) - f_2(x, y_2)| \le \mathcal{L}|y_1 - y_2|, \; y_1, y_2 \in \mathbb{R}.$$

Therefore,

$$\|\mathcal{N}_2 y_1 - \mathcal{N}_2 y_2\| \le \mathcal{L}\|y_1 - y_2\| \sum_{k=0}^{\infty} \|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+\gamma+1)} + \frac{\mathcal{L}}{\Gamma(\gamma+1)} \|y_1 - y_2\| \sum_{k=0}^{\infty} \|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+1)}$$
$$= (\mathcal{L} E_{\beta+\gamma,\gamma+1}(\|\alpha\|) + \frac{\mathcal{L}}{\Gamma(\gamma+1)} E_{\beta+\gamma,1}(\|\alpha\|)) \|y_1 - y_2\| = \mathcal{Q}\|y_1 - y_2\|,$$

which claims that $\mathcal{N}_2$ is a contraction as $\mathcal{Q} < 1$.

As for $\mathcal{N}_1$, we are going to show that (i) $\mathcal{N}_1$ is continuous from $C[0, 1]$ to itself. In fact for $y_1, y_2 \in C[0, 1]$,

$$\mathcal{N}_1 y_1(x) - \mathcal{N}_1 y_2(x) = \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma (f_1(x, y_1(x)) - f_1(x, y_2(x)))$$
$$+ (\chi(y_1) - \chi(y_2)) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x$$
$$+ I^\gamma_{x=1}\alpha(x) I^\beta (y_1(x) - y_2(x)) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x$$
$$- I^\gamma_{x=1}(f_1(x, y_1(x)) - f_1(x, y_2(x))) \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x,$$

which implies that

$$\|\mathscr{N}_1 y_1-\mathscr{N}_1 y_2\|$$
$$\leq \sum_{k=0}^{\infty}\|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+\gamma+1)} \sup_{x\in[0,1]} |f_1(x, y_1(x))-f_1(x, y_2(x))|$$
$$+\mathscr{L}_1\|y_1-y_2\|\sum_{k=0}^{\infty}\|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+1)}+\frac{\|\alpha\|}{\Gamma(\beta+\gamma+1)}\|y_1-y_2\|\sum_{k=0}^{\infty}\|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+1)}$$
$$+\frac{1}{\Gamma(\gamma+1)}\sup_{x\in[0,1]}|f_1(x, y_1(x))-f_1(x, y_2(x))|\sum_{k=0}^{\infty}\|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+1)}$$
$$=\sup_{x\in[0,1]}|f_1(x, y_1(x))-f_1(x, y_2(x))|E_{\beta+\gamma,\gamma+1}(\|\alpha\|)+\mathscr{L}_1\|y_1-y_2\|E_{\beta+\gamma,1}(\|\alpha\|)$$
$$+\frac{\|\alpha\|}{\Gamma(\beta+\gamma+1)}\|y_1-y_2\|E_{\beta+\gamma,1}(\|\alpha\|)+\frac{1}{\Gamma(\gamma+1)}\sup_{x\in[0,1]}|f_1(x, y_1(x))-f_1(x, y_2(x))|E_{\beta+\gamma,1}(\|\alpha\|).$$

Since f_1 is a continuous and bounded function over $[0, 1]\times\mathbb{R}$, we imply $\|\mathscr{N}_1 y_1\| < +\infty$ and $\mathscr{N}_1$ is a continuous mapping from $C[0, 1]$ to itself.

(ii) $\mathscr{N}_1$ maps bounded sets in $C[0, 1]$ to bounded sets. Indeed, let $\mathscr{Y}$ be a bounded set. Then there exists a positive constant $\mathscr{C}$ such that for $y\in\mathscr{Y}$

$$|\chi(y)| = |\chi(y)-\chi(0)+\chi(0)| \leq \mathscr{L}_1\|y\|+|\chi(0)| \leq \mathscr{C}.$$

Thus,

$$\|\mathscr{N}_1 y\| \leq \sup_{x\in[0,1]}|f_1(x, y(x))|E_{\beta+\gamma,\gamma+1}(\|\alpha\|)+\mathscr{C}E_{\beta+\gamma,1}(\|\alpha\|)$$
$$+\frac{\|\alpha\|}{\Gamma(\beta+\gamma+1)}\|y\|E_{\beta+\gamma,1}(\|\alpha\|)++\frac{1}{\Gamma(\gamma+1)}\sup_{x\in[0,1]}|f_1(x, y(x))|E_{\beta+\gamma,1}(\|\alpha\|)$$

is uniformly bounded.

(iii) $\mathscr{N}_1 y$ is equicontinuous over $[0, 1]$ for $y\in\mathscr{Y}$. Let $0\leq x_1 < x_2\leq 1$, then

$$(\mathscr{N}_1 y)(x_2)-(\mathscr{N}_1 y)(x_1)$$
$$= I^{\gamma}_{x=x_2} f_1(x, y(x)) - I^{\gamma}_{x=x_1} f_1(x, y(x)) + \sum_{k=1}^{\infty}(-1)^k(I^{\gamma}_{x=x_2}\phi_k(x) - I^{\gamma}_{x=x_1}\phi_k(x))(= I_1)$$
$$+\chi(y)(x_2-x_1)+\chi(y)\sum_{k=1}^{\infty}(-1)^k(I^{\gamma}_{x=x_2}\psi_k(x)-I^{\gamma}_{x=x_1}\psi_k(x))(= I_2)$$
$$+I^{\gamma}_{x=1}\alpha(x)I^{\beta}y(x)\left[(x_2-x_1)+\sum_{k=1}^{\infty}(-1)^k(I^{\gamma}_{x=x_2}\psi_k(x)-I^{\gamma}_{x=x_1}\psi_k(x))\right](= I_3)$$
$$-I^{\gamma}_{x=1}f_1(x, y(x))\left[(x_2-x_1)+\sum_{k=1}^{\infty}(-1)^k(I^{\gamma}_{x=x_2}\psi_k(x)-I^{\gamma}_{x=x_1}\psi_k(x))\right], (= I_4)$$

where

$$\phi_k(x) = \alpha(x) I^\beta (I^\gamma \alpha(x) I^\beta)^{k-1} I^\gamma f_1(x, y(x)) \quad \text{and} \quad \psi_k(x) = \alpha(x) I^\beta (I^\gamma \alpha(x) I^\beta)^{k-1} x,$$

for all $k \geq 1$. Note that $\chi(y)$, $I^\gamma_{x=1}\alpha(x) I^\beta y(x)$ and $I^\gamma_{x=1} f_1(x, y(x))$ are bounded for $y \in \mathscr{Y}$ since f_1 is bounded, and

$$\|\phi_k\| \leq \|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+1)} \sup_{x\in[0,1]} |f_1(x, y(x))| \quad \text{and}$$
$$\|\psi_k\| \leq \|\alpha\|^k \frac{1}{\Gamma((k-1)(\beta+\gamma)+\beta+1)}.$$

Regarding I_1, the first term is

$$\begin{aligned} &I^\gamma_{x=x_2} f_1(x, y(x)) - I^\gamma_{x=x_1} f_1(x, y(x)) \\ &\quad = \frac{1}{\Gamma(\gamma)} \int_0^{x_1} [(x_2 - x)^{\gamma-1} - (x_1 - x)^{\gamma-1}] f_1(x, y(x)) dx + \frac{1}{\Gamma(\gamma)} \int_{x_1}^{x_2} (x_2 - x)^{\gamma-1} f_1(x, y(x)) dx. \end{aligned}$$

This implies that

$$\begin{aligned} &|I^\gamma_{x=x_2} f_1(x, y(x)) - I^\gamma_{x=x_1} f_1(x, y(x))| \\ &\qquad \leq \frac{1}{\Gamma(\gamma)} \left[\frac{x_2^\gamma}{\gamma} - \frac{x_1^\gamma}{\gamma} \right] \sup_{x\in[0,1]} |f_1(x, y(x))| + \frac{1}{\Gamma(\gamma)} (x_2 - x_1) \sup_{x\in[0,1]} |f_1(x, y(x))| \\ &\qquad \leq \frac{2}{\Gamma(\gamma)} (x_2 - x_1) \sup_{x\in[0,1]} |f_1(x, y(x))|, \end{aligned}$$

by using the mean value theorem

$$0 < \frac{x_2^\gamma}{\gamma} - \frac{x_1^\gamma}{\gamma} \leq x_2 - x_1.$$

The second term

$$\begin{aligned} \left| \sum_{k=1}^\infty (-1)^k (I^\gamma_{x=x_2} \phi_k(x) - I^\gamma_{x=x_1} \phi_k(x)) \right| &\leq \sum_{k=1}^\infty |(I^\gamma_{x=x_2} \phi_k(x) - I^\gamma_{x=x_1} \phi_k(x))| \\ &\leq \frac{2}{\Gamma(\gamma)} (x_2 - x_1) \sum_{k=1}^\infty \|\alpha\|^k \frac{1}{\Gamma(k(\beta+\gamma)+1)} \sup_{x\in[0,1]} |f_1(x, y(x))|. \end{aligned}$$

In summary, I_1 is equicontinuous over [0, 1] for $y \in \mathscr{Y}$ due to the factor $x_2 - x_1$ (clearly I_1 is a family of Lipschitz functions).

As for I_2, the first term

$$|\chi(y)(x_2 - x_1)| \leq \mathscr{C}(x_2 - x_1),$$

and the second term

$$\left|\chi(y)\sum_{k=1}^{\infty}(-1)^k(I^{\gamma}_{x=x_2}\psi_k(x)-I^{\gamma}_{x=x_1}\psi_k(x))\right| \le \mathscr{C}\sum_{k=1}^{\infty}|I^{\gamma}_{x=x_2}\psi_k(x)-I^{\gamma}_{x=x_1}\psi_k(x)|$$
$$\le \frac{2\mathscr{C}}{\Gamma(\gamma)}(x_2-x_1)\sum_{k=1}^{\infty}\|\alpha\|^k\frac{1}{\Gamma((k-1)(\beta+\gamma)+\beta+1)},$$

which infers that I_2 is equicontinuous over [0, 1] for $y \in \mathscr{Y}$. I_3 and I_4 follow similarly. Hence, $\mathscr{N}_1 y$ is equicontinuous over [0, 1]. By the Arzela–Ascoli theorem, $\mathscr{N}_1$ is compact. So, equation (1-1) has at least one solution using Krasnoselskii's fixed point theorem. This completes the proof. □

Theorem 3. *Let $\alpha \in C[0, 1]$, $1 < \gamma \le 2$, $\beta \ge 0$, f_1 be a continuous and bounded mapping from $[0, 1] \times \mathbb{R}$ to $\mathbb{R}$ and $\chi : C[0, 1] \to \mathbb{R}$ be a functional satisfying the following Lipschitz condition for a nonnegative constant $\mathscr{L}_1$*

$$|\chi(y_1)-\chi(y_2)| \le \mathscr{L}_1\|y_1-y_2\|.$$

Then equation

$$\begin{cases} {}_C D^{\gamma} y(x)+\alpha(x)I^{\beta}y(x)=f_1(x, y(x)), & x \in [0, 1], \\ y(0)=0,\ y(1)=\chi(y), \end{cases}$$

has at least one solution in $C[0, 1]$.

Proof. It follows from Theorem 2 by setting $f_2 = 0$. Therefore $\mathscr{L} = 0$ implies $\mathscr{Q} = 0 < 1$. This completes the proof. □

Example 4. The following fractional nonlinear integro-differential equation with variable coefficients and a functional boundary condition:

$$\begin{cases} {}_C D^{1.7} y(x)+\frac{1}{19(1+x^3)}I^{1.2}y(x)=\cos(xy^2(x))+\frac{1}{10(1+x^2)}|x^2+y(x)|, & x \in [0, 1], \\ y(0)=0,\ y(1)=9\int_0^{1/2}|xy(x)|dx, \end{cases}$$

has at least one solution in $C[0, 1]$.

Proof. Clearly,

$$\gamma = 1.7, \quad \beta = 1.2, \quad \|\alpha\| = \tfrac{1}{19} \quad \text{and} \quad f_1(x, y) = \cos(xy^2)$$

is a continuous and bounded function over $[0, 1] \times \mathbb{R}$. Moreover,

$$f_2(x, y) = \frac{1}{10(1+x^2)}|x^2+y|$$

is a Lipschitz function satisfying

$$|f_2(x, y_1)-f_2(x, y_2)| \le \frac{1}{10(1+x^2)}\big||x^2+y_1|-|x^2+y_2|\big| \le \frac{1}{10}|y_1-y_2|,$$

which claims that $\mathscr{L} = \frac{1}{10}$, and

$$y(1) = \chi(y) = 9\int_0^{1/2}|xy(x)|dx$$

is a functional satisfying

$$|\chi(y_1) - \chi(y_2)| \le 9\Big|\int_0^{1/2} |xy_1(x)|dx - \int_0^{1/2} |xy_2(x)|dx\Big| \le \tfrac{9}{4}\|y_1 - y_2\|.$$

Finally we need to evaluate the value of

$$\begin{aligned}\mathscr{Q} &= \mathscr{L} E_{\beta+\gamma,\gamma+1}(\|\alpha\|) + \frac{\mathscr{L}}{\Gamma(\gamma+1)} E_{\beta+\gamma,1}(\|\alpha\|)\\ &= \tfrac{1}{10} E_{2.9,2.7}\big(\tfrac{1}{19}\big) + \frac{1}{10\Gamma(2.7)} E_{2.9,1}\big(\tfrac{1}{19}\big)\\ &\approx \tfrac{1}{10} * 0.648236 + 0.0647381 * 1.00994 = 0.130205196714 < 1.\end{aligned}$$

So the equation has a solution in $C[0, 1]$ from Theorem 2. This completes the proof. □

Example 5. The following fractional nonlinear integro-differential equation:

$$\begin{cases} {}_C D^{1.9} y(x) - 12(\cos x^2 + x^3) I^{3.1} y(x) = \frac{1}{y^2+1+\arctan|y|+x}, & x \in [0, 1],\\ y(0) = 0,\ y(1) = \sin y(0) + \sin y\big(\tfrac{1}{2}\big) + \sin y(1), \end{cases}$$

has at least one solution in $C[0, 1]$.

Proof. Clearly,

$$f_1(x, y) = \frac{1}{y^2 + 1 + \arctan|y| + x}, \quad x \in [0, 1]$$

is a continuous and bounded mapping from $[0, 1] \times \mathbb{R}$ to $\mathbb{R}$, and

$$\alpha(x) = -12(\cos x^2 + x^3)$$

is continuous over $[0, 1]$. Furthermore,

$$\chi(y) = \sin y(0) + \sin y\big(\tfrac{1}{2}\big) + \sin y(1)$$

is a Lipschitz function satisfying

$$|\chi(y_1) - \chi(y_2)| \le |\sin y_1(0) - \sin y_2(0)| + \big|\sin y_1\big(\tfrac{1}{2}\big) - \sin y_2\big(\tfrac{1}{2}\big)\big| + |\sin y_1(1) - \sin y_2(1)| \le 3\|y_1 - y_2\|.$$

By using Theorem 3, the equation has at least one solution. This completes the proof. □

3. Uniqueness

We are going to provide the uniqueness result by using Banach's contractive principle in this section.

Theorem 6. *Let $\alpha \in C[0, 1]$, $1 < \gamma \le 2$, $\beta \ge 0$, f_1 and f_2 be continuous functions satisfying the following Lipschitz conditions for nonnegative constants $\mathscr{C}_1$ and $\mathscr{C}_2$:*

$$\begin{aligned} |f_1(x, y_1) - f_1(x, y_2)| &\le \mathscr{C}_1 |y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R},\\ |f_2(x, y_1) - f_2(x, y_2)| &\le \mathscr{C}_2 |y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}. \end{aligned}$$

Furthermore, we assume $\chi : C[0, 1] \to \mathbb{R}$ *is a functional satisfying the inequality below for a nonnegative constant* $\mathscr{L}$

$$|\chi(y_1) - \chi(y_2)| \le \mathscr{L}\|y_1 - y_2\|, \quad y_1, y_2 \in \mathbb{R},$$

and

$$q = 2(\mathscr{C}_1 + \mathscr{C}_2)E_{\beta+\gamma,\gamma+1}(\|\alpha\|) + \left(\mathscr{L} + \frac{\|\alpha\|}{\Gamma(\gamma+\beta+1)}\right)E_{\beta+\gamma,1}(\|\alpha\|) < 1.$$

Then equation (1-1) *has a unique solution in* $C[0, 1]$.

Proof. We begin by defining a nonlinear mapping M over $C[0, 1]$ as

$$\begin{aligned}(My)(x) = {} & \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma f_1(x, y(x)) + \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma f_2(x, y(x)) \\ & + \chi(y)\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x + I^\gamma_{x=1}\alpha(x) I^\beta y(x)\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x \\ & - I^\gamma_{x=1} f_1(x, y(x))\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x \\ & - I^\gamma_{x=1} f_2(x, y(x))\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x.\end{aligned}$$

Noting that

$$|f_1(x, y(x))| = |f_1(x, y(x)) - f_1(x, 0) + f_1(x, 0)| \le \mathscr{C}_1\|y\| + \max_{x\in[0,1]}|f_1(x, 0)| < +\infty,$$

and the proof of Theorem 2, we claim that M is a mapping from $C[0, 1]$ to itself. It remains to be shown that M is contractive. In fact,

$$\begin{aligned}(My_1)(x) - (My_2)(x) = {} & \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma (f_1(x, y_1(x)) - f_1(x, y_2(x))) \\ & + \sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k I^\gamma (f_2(x, y_1(x)) - f_2(x, y_2(x))) \\ & + (\chi(y_1) - \chi(y_2))\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x \\ & + I^\gamma_{x=1}\alpha(x) I^\beta (y_1(x) - y_2(x))\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x \\ & - I^\gamma_{x=1}(f_1(x, y_1(x)) - f_1(x, y_2(x)))\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x \\ & - I^\gamma_{x=1}(f_2(x, y_1(x)) - f_2(x, y_2(x)))\sum_{k=0}^{\infty}(-1)^k (I^\gamma \alpha(x) I^\beta)^k x.\end{aligned}$$

This implies that

$$\begin{aligned}\|My_1 - My_2\| &\le 2\mathscr{C}_1 \sum_{k=1}^{\infty} \frac{\|\alpha\|^k}{\Gamma((\gamma+\beta)k+\gamma+1)} \|y_1 - y_2\| \\ &\quad + 2\mathscr{C}_2 \sum_{k=1}^{\infty} \frac{\|\alpha\|^k}{\Gamma((\gamma+\beta)k+\gamma+1)} \|y_1 - y_2\| + \mathscr{L} \sum_{k=1}^{\infty} \frac{\|\alpha\|^k}{\Gamma((\gamma+\beta)k+1)} \|y_1 - y_2\| \\ &\quad + \frac{\|\alpha\|}{\Gamma(\gamma+\beta+1)} \sum_{k=0}^{\infty} \frac{\|\alpha\|^k}{\Gamma((\gamma+\beta)k+1)} \|y_1 - y_2\| = q\|y_1 - y_2\|.\end{aligned}$$

Since $q < 1$, equation (1-1) has a unique solution by Banach's contractive principle. We complete the proof. □

Example 7. The following fractional nonlinear integro-differential equation:

$$\begin{cases} {}_CD^{1.5}y(x) + \frac{1}{23(1+x)} I^{1.6}y(x) = \frac{1}{17}\cos(x^4 + |y(x)|) + \frac{1}{10(1+x^2)} \arctan y(x), \\ x \in [0, 1], \\ y(0) = 0,\ y(1) = \frac{1}{11}\int_0^1 |x^7 + y(x)|dx, \end{cases}$$

has a unique solution in $C[0, 1]$.

Proof. Clearly,

$$f_1(x, y) = \tfrac{1}{17}\cos(x^4 + |y|) \quad \text{and} \quad f_2(x, y) = \frac{1}{10(1+x^2)} \arctan y$$

are Lipschitz functions satisfying

$$\begin{aligned}|f_1(x, y_1) - f_1(x, y_2)| &\le \tfrac{1}{17}|\cos(x^4 + |y_1|) - \cos(x^4 + |y_2|)| \\ &\le \tfrac{1}{17}|(x^4 + |y_1|) - (x^4 + |y_2|)| \le \tfrac{1}{17}|y_1 - y_2|,\end{aligned}$$

and

$$|f_2(x, y_1) - f_2(x, y_2)| \le \tfrac{1}{10}|y_1 - y_2|.$$

Hence $\mathscr{C}_1 = \frac{1}{17}$ and $\mathscr{C}_2 = \frac{1}{10}$. Furthermore,

$$\chi(y) = \tfrac{1}{11}\int_0^1 |x^7 + y(x)|dx$$

satisfies

$$|\chi(y_1) - \chi(y_2)| \le \tfrac{1}{11}\|y_1 - y_2\|,$$

which claims that $\mathscr{L} = \frac{1}{11}$. It follows from the equation that

$$\gamma = 1.5,\ \beta = 1.6,\ \|\alpha\| = \tfrac{1}{23}.$$

Finally, we need to evaluate the value of q:

$$\begin{aligned} q &= 2(\mathscr{C}_1 + \mathscr{C}_2) E_{\beta+\gamma,\gamma+1}(\|\alpha\|) + \left(\mathscr{L} + \frac{\|\alpha\|}{\Gamma(\gamma+\beta+1)}\right) E_{\beta+\gamma,1}(\|\alpha\|) \\ &= 2\left(\tfrac{1}{17} + \tfrac{1}{10}\right) E_{3.1,2.5}\left(\tfrac{1}{23}\right) + \left(\frac{1}{11} + \frac{1}{23\Gamma(4.1)}\right) E_{3.1,1}\left(\tfrac{1}{23}\right) \\ &\approx 2\left(\tfrac{1}{17} + \tfrac{1}{10}\right) * 0.752959 + \left(\frac{1}{11} + \frac{1}{23\Gamma(4.1)}\right) * 1.00638 \\ &= 0.239175 + 0.0979118 < 1. \end{aligned}$$

By Theorem 6, the equation has a unique solution. This completes the proof. □

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