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Existence, Uniqueness, and Hyers–Ulam's Stability of the Nonlinear Bagley–Torvik Equation with Functional Initial Conditions

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Abstract

The nonlinear Bagley–Torvik equation is of fundamental importance, as it captures a realistic and intricate interplay among memory effects, nonlinearity, and functional dependence—making it a powerful model for a wide range of natural and engineered systems. Its analysis contributes significantly to both the theoretical development of fractional differential equations and their practical applications across science and technology. In this paper, we employ the inverse operator method, the multivariate Mittag-Leffler function, and several classical fixed-point theorems to establish sufficient conditions for the existence, uniqueness, and Hyers–Ulam stability of solutions to the nonlinear Bagley–Torvik equation with functional initial conditions. Finally, we present several examples by explicitly computing values of the multivariate Mittag-Leffler functions to illustrate the main results.

Keywords: nonlinear Bagley–Torvik equation; uniqueness and existence; Hyers–Ulam's stability; fixed-point theory; multivariate Mittag-Leffler function; inverse operator

MSC: 34B15; 34A12; 34K20; 26A33

1. Introduction

The Riemann–Liouville fractional integral I^α of order $\alpha \in \mathbb{R}^+$ is defined for the function $y \in C[0,T]$ with $T > 0$ [1,2] as

$$(I^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} y(\sigma) d\sigma, \quad t \in [0,T].$$

It follows that

$$I^0 y = y,$$

by noting that

$$\delta(t-\sigma) = \frac{(t-\sigma)_+^{-1}}{\Gamma(0)}$$

in the distributional sense from refs. [3,4].

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The Liouville–Caputo fractional derivative ${}_CD^{\beta}$ of order $\beta \in (1,2]$ of the function $y \in C^2[0,T]$ is defined as ref. [1]

$$({}_CD^{\beta}y)(t) = (I^{2-\beta}\frac{d^2}{dt^2}y)(t) = \frac{1}{\Gamma(2-\beta)}\int_0^t (t-\sigma)^{1-\beta}y''(\sigma)d\sigma.$$

Thus

$$I^{\beta}({}_CD^{\beta}y)(t) = y(t) - y(0) - y'(0)t.$$

The set $C[0,T]$ is a Banach space consisting of all continuous functions from $[0,T]$ into $\mathbb{R}$, endowed with the norm

$$\|y\| = \max_{t\in[0,T]} |y(t)|.$$

Let ϕ_1 and ϕ_2 be functionals from $C[0,T]$ to $\mathbb{R}$. The objective of this work is to investigate the existence, uniqueness, and Hyers–Ulam stability of solutions to the following nonlinear Bagley–Torvik equation with functional initial conditions:

$$\begin{cases} y^{(2)}(t) + b\,{}_cD^{3/2}y(t) + c\,y(t) = g(t,y(t)), \quad t \in [0,T], \\ y(0) = \phi_1(y), \quad y'(0) = \phi_2(y). \end{cases} \tag{1}$$

where b and c are constants.

The Caputo fractional derivative of order $3/2$ naturally arises when modeling hereditary damping that lies between purely viscous damping (first-order derivative) and elastic inertia effects (second-order derivative). In particular, experimental observations in viscoelastic materials show that damping forces often follow a power-law memory kernel, which leads mathematically to fractional derivatives of order $1 < \alpha < 2$, with $\alpha = 3/2$ being a canonical and physically meaningful choice. Furthermore, the Caputo fractional derivative allows initial conditions to be stated in terms of integer-order derivatives, such as $y(0)$ and $y'(0)$, which are natural in physical and engineering models.

The key technique employed in this paper is to transform the above problem into an equivalent implicit integral equation by the Riemann–Liouville fractional integral. A suitable nonlinear operator is then constructed, and the existence, uniqueness, and Hyers–Ulam stability of solutions are established by applying several well-known fixed-point theorems.

The equation presented above—the nonlinear Bagley–Torvik equation with functional initial conditions—is important for both theoretical and applied reasons:

(1) The equation generalizes the classical Bagley–Torvik model, which was originally introduced to describe damping in viscoelastic systems. By incorporating fractional derivatives and nonlinear terms, our version captures more complex physical behaviors that are not modeled by integer-order differential equations.

(2) The presence of the Caputo fractional derivative of order $3/2$ introduces memory effects, meaning the system's current state depends on its entire history. This is essential for modeling viscoelastic materials, anomalous diffusion, complex biological or physiological systems, and nonlocal phenomena in physics and engineering.

(3) The use of functional conditions—where initial values depend on the entire solution function $y(t)$, not just fixed numbers—makes the problem highly relevant in control systems with feedback mechanisms, population dynamics with state-dependent delays, and systems with aftereffects, like materials with memory or economic models with cumulative influence. This reflects real-world processes better than classical pointwise initial conditions.

(4) The problem is mathematically rich and challenging, combining nonlinearity, fractional calculus, and functional analysis. It serves as testbeds for developing and applying

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advanced tools like fixed-point theorems, Mittag-Leffler functions, and operator theory. Studying existence, uniqueness, and Hyers–Ulam's stability helps establish the well-posedness of such equation, which is essential before attempting numerical solutions or physical interpretations.

(5) The equation can be applied to model and analyze vibration of viscoelastic plates and beams, biomechanical systems involving hereditary properties, and control and feedback systems in engineering and signal processing involving memory or fractional dynamics.

Let $\alpha_1, \cdots, \alpha_m, \beta > 0$ and $z_1, \cdots, z_m \in \mathbb{C}$. Then

$$\begin{aligned} &E_{(\alpha_1,\cdots,\alpha_m),\,\beta}(z_1,\cdots,z_m) \\ &\quad = \sum_{s=0}^{\infty} \sum_{s_1+\cdots+s_m=s} \binom{s}{s_1,\cdots,s_m} \frac{z_1^{s_1}\cdots z_m^{s_m}}{\Gamma(\alpha_1 s_1+\cdots+\alpha_m s_m+\beta)} \\ &\quad = \sum_{s=0}^{\infty} \sum_{s_1+\cdots+s_m=s} \frac{s!}{s_1!\cdots s_m!} \frac{z_1^{s_1}\cdots z_m^{s_m}}{\Gamma(\alpha_1 s_1+\cdots+\alpha_m s_m+\beta)} \end{aligned}$$

is the well-known multivariate Mittag-Leffler function [1,5], which is an entire function on complex plane $\mathbb{C}^m$. When $m = 1$, it reduces to the following two-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(\alpha s+\beta)}, \quad \alpha, \beta > 0, \; z \in \mathbb{C}.$$

If $\beta = 1$, we obtain the classical Mittag-Leffler function defined by

$$E_{\alpha}(z) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(\alpha s+1)}, \quad \alpha > 0, \; z \in \mathbb{C}.$$

The inverse operator method is a powerful and effective tool for handling fractional differential equations, including fractional PDEs, under a variety of initial and boundary value conditions [6,7]. To demonstrate the use of the inverse operator technique, we begin by finding a solution to the following Bagley–Torvik equation for $f \in C[0,T]$ and $y \in C^2[0,T]$:

$$\begin{cases} y^{(2)}(t) + b\ {}_cD^{3/2}y(t) + c\ y(t) = f(t), \\ y(0) = y_0, \;\; y'(0) = y_1, \end{cases} \tag{2}$$

where $y_0, y_1 \in \mathbb{R}$.

Clearly,

$$I^2 y^{(2)}(t) = y(t) - y(0) - y'(0)t = y(t) - y_0 - y_1 t,$$

by noting that $y \in C^2[0,T]$. Applying the operator I^2 to Equation (2) yields

$$y(t) - y_0 - y_1 t + b\ I^{1/2}(I^{3/2}{}_cD^{3/2})y(t) + c\ I^2 y(t) = I^2 f(t).$$

It follows that

$$I^{1/2}(I^{3/2}{}_cD^{3/2})y(t) = I^{1/2}(y(t) - y_0 - y_1 t) = I^{1/2}y(t) - \frac{y_0 t^{1/2}}{\Gamma(\frac{1}{2}+1)} - \frac{y_1 t^{3/2}}{\Gamma(\frac{1}{2}+2)}.$$

Thus,

$$\left(1+b\ I^{1/2}+c\ I^2\right)y(t)=I^2f(t)+y_0+y_1t+\frac{by_0t^{1/2}}{\Gamma(\frac{1}{2}+1)}+\frac{by_1t^{3/2}}{\Gamma(\frac{1}{2}+2)}. \tag{3}$$

We are going to show that a unique inverse operator of $1+b\ I^{1/2}+c\ I^2$ is

$$P=\sum_{s=0}^{\infty}(-1)^s\left(b\ I^{1/2}+c\ I^2\right)^s=\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}(-1)^s\binom{s}{s_1,s_2}b^{s_1}c^{s_2}I^{\frac{1}{2}s_1+2s_2} \tag{4}$$

in the space $C[0,T]$. Indeed, for any $\phi\in C[0,T]$, we have

$$\begin{aligned}\|P\phi\|&\le\|\phi\|\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}\binom{s}{s_1,s_2}|b|^{s_1}|c|^{s_2}\left\|I^{\frac{1}{2}s_1+2s_2}\right\|\\&\le\|\phi\|\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}\binom{s}{s_1,s_2}|b|^{s_1}|c|^{s_2}\frac{T^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1+2s_2+1)}\\&=\|\phi\|E_{(1/2,2),1}\left(|b|T^{1/2},|c|T^2\right)<+\infty,\end{aligned}$$

which implies that P is a mapping from $C[0,T]$ to itself. Furthermore,

$$P\left(1+b\ I^{1/2}+c\ I^2\right)=\left(1+b\ I^{1/2}+c\ I^2\right)P=1\ \ \text{(identity)}.$$

In fact,

$$\begin{aligned}P\left(1+b\ I^{1/2}+c\ I^2\right)&=P+\sum_{s=0}^{\infty}(-1)^s\left(b\ I^{1/2}+c\ I^2\right)^{s+1}\\&=1+\sum_{s=1}^{\infty}(-1)^s\left(b\ I^{1/2}+c\ I^2\right)^s+\sum_{s=0}^{\infty}(-1)^s\left(b\ I^{1/2}+c\ I^2\right)^{s+1}\\&=1.\end{aligned}$$

Similarly,

$$\left(1+b\ I^{1/2}+c\ I^2\right)P=1$$

and the uniqueness of P follows easily.

From Equation (3), we get

$$\begin{aligned}y(t)&=P\left(I^2f(t)+y_0+y_1t+\frac{by_0t^{1/2}}{\Gamma(\frac{1}{2}+1)}+\frac{by_1t^{3/2}}{\Gamma(\frac{1}{2}+2)}\right)\\&=\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}(-1)^s\binom{s}{s_1,s_2}b^{s_1}c^{s_2}I^{\frac{1}{2}s_1+2s_2+2}f(t)\\&\quad+y_0\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}(-1)^s\binom{s}{s_1,s_2}b^{s_1}c^{s_2}\frac{t^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1+2s_2+1)}\\&\quad+y_1\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}(-1)^s\binom{s}{s_1,s_2}b^{s_1}c^{s_2}\frac{t^{\frac{1}{2}s_1+2s_2+1}}{\Gamma(\frac{1}{2}s_1+2s_2+2)}\\&\quad+b\,y_0\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}(-1)^s\binom{s}{s_1,s_2}b^{s_1}c^{s_2}\frac{t^{\frac{1}{2}s_1+2s_2+\frac{1}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{3}{2})}\\&\quad+b\,y_1\sum_{s=0}^{\infty}\sum_{s_1+s_2=s}(-1)^s\binom{s}{s_1,s_2}b^{s_1}c^{s_2}\frac{t^{\frac{1}{2}s_1+2s_2+\frac{3}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{5}{2})},\end{aligned} \tag{5}$$

using

$$I^{\gamma} t^{\gamma_0} = \frac{\Gamma(\gamma_0 + 1)}{\Gamma(\gamma + \gamma_0 + 1)} t^{\gamma + \gamma_0},$$

for all $\gamma \geq 0$ and $\gamma_0 > -1$. Clearly, y is in $C^2[0, T]$ with $y(0) = y_0$ and $y'(0) = y_1$ by observing that the sum

$$\begin{aligned} S_0(t) = y_0 \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1 + 2s_2}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 1)} \\ + b\, y_0 \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1 + 2s_2 + \frac{1}{2}}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + \frac{3}{2})} \end{aligned}$$

is an element in $C^2[0, T]$ and $S_0'(0) = 0$ due to cancellations. To address these in detail, we see that the first sum:

$$y_0 \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1 + 2s_2}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 1)}$$

contributes the coefficient $-y_0 b \dfrac{1}{\Gamma(3/2)}$ for the term $t^{1/2}$ and the second sum:

$$b\, y_0 \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1 + 2s_2 + \frac{1}{2}}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + \frac{3}{2})}$$

adds $y_0 b \dfrac{t^{1/2}}{\Gamma(3/2)}$, which cancels each other. Regarding t and $t^{3/2}$, the same cases follow. Hence,

$$S_0(t) = y_0 + a_2 t^2 + \cdots = y_0 + O(t^2),$$

which belongs to $C^2[0, T]$. Furthermore, the sum

$$\begin{aligned} S_1(t) = y_1 \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1 + 2s_2 + 1}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 2)} \\ + b\, y_1 \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1 + 2s_2 + \frac{3}{2}}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + \frac{5}{2})} \\ = y_1 t + O(t^3) \end{aligned}$$

satisfies $S_1'(0) = y_1$ and $S_1(t) \in C^2[0, T]$ by cancellations. In addition, the first term

$$\sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} I^{\frac{1}{2}s_1 + 2s_2 + 2} f(t)$$

is obviously in $C^2[0, T]$ because of the factor I^2 and f being continuous over $[0, T]$. The uniqueness of solutions follows immediately from the fact that equation

$$\begin{cases} y^{(2)}(t) + b\ {}_cD^{3/2} y(t) + c\, y(t) = 0, \\ y(0) = 0, \quad y'(0) = 0, \end{cases}$$

only has solution zero.

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Remark 1. *Indeed, we can write $S_0(t)$ as*

$$S_0(t) = y_0 \sum_{\nu \in \mathcal{I}} a_\nu t^\nu,$$

where $\mathcal{I} = \left\{ \frac{1}{2}s_1 + 2s_2, \frac{1}{2}s_1 + 2s_2 + \frac{1}{2} : s_1, s_2 \in \mathbb{N}_0 \right\}$. *The coefficient a_ν of t^ν is*

$$\begin{aligned} a_\nu = &\sum_{s_1,s_2 \geq 0,\ \frac{1}{2}s_1+2s_2=\nu} (-1)^{s_1+s_2} \binom{s_1+s_2}{s_1} b^{s_1} c^{s_2} \frac{1}{\Gamma(\nu+1)} \\ &+ \sum_{s_1,s_2 \geq 0,\ \frac{1}{2}s_1+2s_2+\frac{1}{2}=\nu} (-1)^{s_1+s_2} \binom{s_1+s_2}{s_1} b^{s_1+1} c^{s_2} \frac{1}{\Gamma((\nu-\frac{1}{2})+\frac{3}{2})}, \end{aligned}$$

with values

$$a_0 = y_0, \quad a_{1/2} = a_1 = a_{3/2} = 0, \quad a_2 = -\frac{cy_0}{2}.$$

The Bagley–Torvik equation was first introduced by Bagley and Torvik [8,9] in the context of modeling the oscillatory motion of a rigid body in a Newtonian fluid, where the damping term is more accurately captured by a fractional derivative of order $3/2$ rather than an integer order.

There has been extensive research on the Bagley–Torvik equation, extending it to nonlinear forms, generalized fractional orders, and complex boundary conditions. Podlubny [10] laid the theoretical foundation for fractional differential equations, including the Caputo and Riemann–Liouville definitions, which are crucial for interpreting and solving the Bagley–Torvik equation using tools such as Mittag–Leffler functions and Laplace transforms. Kilbas et al. [1] investigated existence and uniqueness results for fractional and integro-differential equations [11] using fixed-point theorems, which have been applied to nonlinear variants of the Bagley–Torvik equation. In 2015, Labecca et al. [12] obtained a general analytical solution of Bagley-Torvik equation defined by Caputo's fractional derivatives in terms of Wiman's functions and their derivatives. The reasoning was conducted by using Laplace transform and Mittag-Leffler functions.

Diethelm [13] provided numerical methods for solving fractional differential equations, including predictor–corrector schemes that are directly applicable to Bagley–Torvik-type models. Aljazzazi et al. [14] investigated the performance of reproducing kernel Hilbert space method to approximately solve a class of fractional Bagley-Torvik equations fitted with an integral boundary condition of the form by numerical analysis:

$$\begin{cases} y^{(2)}(t) + p(t)\,{}_cD^\alpha y(t) + q(t)\,y(t) = g(t, y(t)), \quad t \in [a,b], 1 < \alpha \leq 2, \\ y(a) = y_0, \ \ y(b) = \lambda I^\beta y(b), \quad 1 < \beta \leq 2, \end{cases}$$

where y_0 and λ are real constants, $p, q,$ and g are the known functions. In addition, Yüzbaşi [15] used the collocation points, the matrix operations and a generalization of the Bessel functions of the first kind, and presented the approximate solution of the Bagley–Torvik equation with variable coefficients and boundary conditions.

The study of Hyers–Ulam stability originates from a fundamental question in the theory of functional equations and has significant implications across analysis, differential equations, and applied mathematics [16,17]. The central idea is as follows:

If a function approximately satisfies a functional equation, can it be approximated by an exact solution of that equation?

In real applications (physics, engineering, finance), models are never exact. Perturbed equations arise due to: measurement errors, numerical approximations, modeling assumptions. Hyers–Ulam's stability tells us whether small deviations in equations lead to small deviations in solutions — a kind of robustness guarantee.

Definition 1. *Equation (1) is said to be Hyers-Ulam stable if there exists a constant $K > 0$ such that for each $\epsilon > 0$ and any approximate solution $y \in C^2[0, T]$ satisfying*

$$\left\| y^{(2)}(t) + b\,{}_cD^{3/2}y(t) + c\,y(t) - g(t, y(t)) \right\| \leq \epsilon,$$

there exists an exact solution $y_0 \in C^2[0, T]$ of Equation (1) satisfying

$$\|y - y_0\| \leq K\epsilon,$$

with the initial conditions:

$$y_0(0) = y(0), \quad y_0'(0) = y'(0),$$

where K is independent of y, y_0 and ϵ.

In other words, if one has an approximate solution to Equation (1) that is close to being an exact solution, then there is an exact solution to the approximate solution. The constant K determines how close the exact solution is to the approximate solution, based on the level of approximation.

The remainder of the paper is organized as follows. In Section 2, we establish sufficient conditions for the uniqueness of solutions to Equation (1), employing a nonlinear operator, the multivariate Mittag-Leffler function, and Banach's fixed-point theorem. In Section 3, we address the existence of solutions using the Leray–Schauder fixed-point theorem and the Arzelà–Ascoli theorem. Section 4 is devoted to the analysis of Hyers–Ulam stability, with a supporting example. Section 5 includes several illustrative examples demonstrating the applicability of the main results. Finally, Section 6 presents a concise summary of the study.

2. Uniqueness

The following theorem deals with the uniqueness for Equation (1).

Theorem 1. *Let g be a continuous and bounded function on $[0, T] \times \mathbb{R}$, satisfying the Lipschitz condition for $\Omega_0 \geq 0$:*

$$|g(t, y_1) - g(t, y_2)| \leq \Omega_0 |y_1 - y_2|, \quad \text{for all } t \in [0, T], \quad y_1,\ y_2 \in \mathbb{R}.$$

Let ϕ_1 and ϕ_2 be functionals from $C[0, T]$ to $\mathbb{R}$ that satisfy the conditions for $\Omega_1,\ \Omega_2 \geq 0$:

$$|\phi_1(y_1) - \phi_1(y_2)| \leq \Omega_1 \|y_1 - y_2\|,$$
$$|\phi_2(y_1) - \phi_2(y_2)| \leq \Omega_2 \|y_1 - y_2\|,$$

where for all y_1, $y_2 \in C[0, T]$. Assume further that

$$\begin{aligned} B = {} & \Omega_0 T^2 E_{(1/2,2),\,3}\left(|b|T^{1/2},\ |c|T^2\right) + \Omega_1 E_{(1/2,2),\,1}\left(|b|T^{1/2},\ |c|T^2\right) \\ & + \Omega_2 T E_{(1/2,2),\,2}\left(|b|T^{1/2},\ |c|T^2\right) + |b|\Omega_1 T^{1/2} E_{(1/2,2),\,3/2}\left(|b|T^{1/2},\ |c|T^2\right) \\ & + |b|\Omega_2 T^{3/2} E_{(1/2,2),\,5/2}\left(|b|T^{1/2},\ |c|T^2\right) < 1. \end{aligned}$$

Then Equation (1) has a unique solution.

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Proof. We begin by defining a nonlinear operator $\mathcal{M}$ over the space $C[0,T]$ from Equation (5) as

$$(\mathcal{M}y)(t) = \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} I^{\frac{1}{2}s_1+2s_2+2} g(t, y(t))$$
$$+ \phi_1(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1+2s_2+1)}$$
$$+ \phi_2(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+1}}{\Gamma(\frac{1}{2}s_1+2s_2+2)}$$
$$+ b\,\phi_1(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{1}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{3}{2})}$$
$$+ b\,\phi_2(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{3}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{5}{2})}.$$

Clearly for any $y \in C[0,T]$,

$$|\phi_1(y)| = |\phi_1(y) - \phi_1(0) + \phi_1(0)| \leq \Omega_1 \|y\| + |\phi_1(0)| < +\infty.$$

Similarly, $|\phi_2(y)| < +\infty$ if $y \in C[0,T]$. Let

$$\|g\| = \sup_{(t,y)\in[0,T]\times\mathbb{R}} |g(t,y)| < +\infty,$$

by noting that g is bounded. Thus for $y \in C[0,T]$,

$$\|\mathcal{M}y\| \leq \|g\| \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} \binom{s}{s_1, s_2} |b|^{s_1} |c|^{s_2} \frac{T^{\frac{1}{2}s_1+2s_2+2}}{\Gamma(\frac{1}{2}s_1+2s_2+3)}$$
$$+ |\phi_1(y)| \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} \binom{s}{s_1, s_2} |b|^{s_1} |c|^{s_2} \frac{T^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1+2s_2+1)}$$
$$+ |\phi_2(y)| \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} \binom{s}{s_1, s_2} |b|^{s_1} |c|^{s_2} \frac{T^{\frac{1}{2}s_1+2s_2+1}}{\Gamma(\frac{1}{2}s_1+2s_2+2)}$$
$$+ |b\phi_1(y)| \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} \binom{s}{s_1, s_2} |b|^{s_1} |c|^{s_2} \frac{T^{\frac{1}{2}s_1+2s_2+\frac{1}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{3}{2})}$$
$$+ |b\phi_2(y)| \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} \binom{s}{s_1, s_2} |b|^{s_1} |c|^{s_2} \frac{T^{\frac{1}{2}s_1+2s_2+\frac{3}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{5}{2})}$$
$$= \|g\| T^2 E_{(1/2,2),\,3}\left(|b|T^{1/2},\ |c|T^2\right) + |\phi_1(y)| E_{(1/2,2),\,1}\left(|b|T^{1/2},\ |c|T^2\right)$$
$$+ |\phi_2(y)| T E_{(1/2,2),\,2}\left(|b|T^{1/2},\ |c|T^2\right) + |b\phi_1(y)| T^{1/2} E_{(1/2,2),\,3/2}\left(|b|T^{1/2},\ |c|T^2\right)$$
$$+ |b\phi_2(y)| T^{3/2} E_{(1/2,2),\,5/2}\left(|b|T^{1/2},\ |c|T^2\right) < +\infty. \tag{6}$$

This implies that $\mathcal{M}$ is a mapping from $C[0,T]$ to itself. It remains to be shown that it is contractive. In fact,

$$\|\mathcal{M}y_1 - \mathcal{M}y_2\| \leq \Omega_0 T^2 E_{(1/2,2),\,3}\left(|b|T^{1/2},\ |c|T^2\right) \|y_1 - y_2\|$$
$$+ \Omega_1 \|y_1 - y_2\| E_{(1/2,2),\,1}\left(|b|T^{1/2},\ |c|T^2\right) + \Omega_2 \|y_1 - y_2\| T E_{(1/2,2),\,2}\left(|b|T^{1/2},\ |c|T^2\right)$$
$$+ |b| \Omega_1 T^{1/2} \|y_1 - y_2\| E_{(1/2,2),\,3/2}\left(|b|T^{1/2},\ |c|T^2\right)$$
$$+ |b| \Omega_2 T^{3/2} \|y_1 - y_2\| E_{(1/2,2),\,5/2}\left(|b|T^{1/2},\ |c|T^2\right) = B \|y_1 - y_2\|,$$

where

$$B = \Omega_0 T^2 E_{(1/2,2),\,3}\Big(|b|T^{1/2},\,|c|T^2\Big) + \Omega_1 E_{(1/2,2),\,1}\Big(|b|T^{1/2},\,|c|T^2\Big)$$
$$+ \Omega_2 T E_{(1/2,2),\,2}\Big(|b|T^{1/2},\,|c|T^2\Big) + |b|\Omega_1 T^{1/2} E_{(1/2,2),\,3/2}\Big(|b|T^{1/2},\,|c|T^2\Big)$$
$$+ |b|\Omega_2 T^{3/2} E_{(1/2,2),\,5/2}\Big(|b|T^{1/2},\,|c|T^2\Big) < 1.$$

By Banach's contractive principle, Equation (1) has a unique solution. This completes the proof. □

3. Existence

Theorem 2. *Assume g is a continuous and bounded function on $[0,T]\times\mathbb{R}$, and ϕ_1 and ϕ_2 are functionals from $C[0,T]$ to $\mathbb{R}$ that satisfy the conditions for $\Omega_1,\ \Omega_2 \ge 0$:*

$$|\phi_1(y_1) - \phi_1(y_2)| \le \Omega_1 \|y_1 - y_2\|,$$
$$|\phi_2(y_1) - \phi_2(y_2)| \le \Omega_2 \|y_1 - y_2\|,$$

where for all $y_1,\ y_2 \in C[0,T]$. In addition,

$$B_0 = 1 - \Omega_1 E_{(1/2,2),\,1}\Big(|b|T^{1/2},\,|c|T^2\Big) - \Omega_2 T E_{(1/2,2),\,2}\Big(|b|T^{1/2},\,|c|T^2\Big)$$
$$- |b|\Omega_1 T^{1/2} E_{(1/2,2),\,3/2}\Big(|b|T^{1/2},\,|c|T^2\Big) - |b|\Omega_2 T^{3/2} E_{(1/2,2),\,5/2}\Big(|b|T^{1/2},\,|c|T^2\Big) > 0.$$

Then there exists at least one solution to Equation (1).

Proof. Again we consider the nonlinear mapping $\mathcal{M}$ over $C[0,T]$:

$$(\mathcal{M}y)(t) = \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1,\,s_2} b^{s_1} c^{s_2} I^{\frac{1}{2}s_1+2s_2+2} g(t, y(t))$$
$$+ \phi_1(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1,\,s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1+2s_2+1)}$$
$$+ \phi_2(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1,\,s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+1}}{\Gamma(\frac{1}{2}s_1+2s_2+2)}$$
$$+ b\,\phi_1(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1,\,s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{1}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{3}{2})}$$
$$+ b\,\phi_2(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1,\,s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{3}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{5}{2})}.$$

It follows from the proof of Theorem 1 that $\mathcal{M}$ is a mapping from $C[0,T]$ to itself. We are going to show that (i) $\mathcal{M}$ is continuous. Indeed,

$$\|\mathcal{M}y_1 - \mathcal{M}y_2\| \le T^2 E_{(1/2,2),\,3}\Big(|b|T^{1/2},\,|c|T^2\Big) \sup_{t\in[0,T]} |g(t,y_1(t)) - g(t,y_2(t))|$$
$$+ \Omega_1 E_{(1/2,2),\,1}\Big(|b|T^{1/2},\,|c|T^2\Big)\|y_1 - y_2\|$$
$$+ \Omega_2 T E_{(1/2,2),\,2}\Big(|b|T^{1/2},\,|c|T^2\Big)\|y_1 - y_2\|$$
$$+ |b|\Omega_1 T^{1/2} E_{(1/2,2),\,3/2}\Big(|b|T^{1/2},\,|c|T^2\Big)\|y_1 - y_2\|$$
$$+ |b|\Omega_2 T^{3/2} E_{(1/2,2),\,5/2}\Big(|b|T^{1/2},\,|c|T^2\Big)\|y_1 - y_2\|,$$

which claims $\mathcal{M}$ is continuous since g is continuous.

(ii) Further, we show that $\mathcal{M}$ is a mapping from a bounded set to a bounded set in $C[0,T]$. Let W be a bounded set in $C[0,T]$. Then $\phi_1(y)$ and $\phi_2(y)$ are bounded for all $y \in W$. It follows from inequality (6) that $\|\mathcal{M}y\|$ is bounded as g is bounded as well.

(iii) We will show that $\mathcal{M}$ is completely continuous from $C[0,T]$ to itself. Then using the Arzela-Ascoli theorem, we only need to prove that $\mathcal{M}$ is equicontinuous on every bounded set W of $C[0,T]$. To proceed this, we let $0 \le t_1 < t_2 \le T$ and $y \in W$, and consider

$$
\begin{aligned}
&(\mathcal{M}y)(t_2) - (\mathcal{M}y)(t_1) \\
&= \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \left(I_{t=t_2}^{\frac{1}{2}s_1+2s_2+2} g(t,y(t)) - I_{t=t_1}^{\frac{1}{2}s_1+2s_2+2} g(t,y(t)) \right) \quad (= I_1) \\
&\quad + \phi_1(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t_2^{\frac{1}{2}s_1+2s_2} - t_1^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1+2s_2+1)} \quad (= I_2) \\
&\quad + \phi_2(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t_2^{\frac{1}{2}s_1+2s_2+1} - t_1^{\frac{1}{2}s_1+2s_2+1}}{\Gamma(\frac{1}{2}s_1+2s_2+2)} \quad (= I_3) \\
&\quad + b\,\phi_1(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t_2^{\frac{1}{2}s_1+2s_2+\frac{1}{2}} - t_1^{\frac{1}{2}s_1+2s_2+\frac{1}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{3}{2})} \quad (= I_4) \\
&\quad + b\,\phi_2(y) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t_2^{\frac{1}{2}s_1+2s_2+\frac{3}{2}} - t_1^{\frac{1}{2}s_1+2s_2+\frac{3}{2}}}{\Gamma(\frac{1}{2}s_1+2s_2+\frac{5}{2})} \quad (= I_5).
\end{aligned}
$$

Regarding I_1, we consider

$$
\begin{aligned}
&I_{t=t_2}^{\frac{1}{2}s_1+2s_2+2} g(t,y(t)) - I_{t=t_1}^{\frac{1}{2}s_1+2s_2+2} g(t,y(t)) \\
&= \frac{1}{\Gamma(\frac{1}{2}s_1+2s_2+2)} \int_0^{t_1} \left((t_2-\tau)^{\frac{1}{2}s_1+2s_2+1} - (t_1-\tau)^{\frac{1}{2}s_1+2s_2+1} \right) g(\tau, y(\tau)) d\tau \quad (= I_{11}) \\
&\quad + \frac{1}{\Gamma(\frac{1}{2}s_1+2s_2+2)} \int_{t_1}^{t_2} (t_2-\tau)^{\frac{1}{2}s_1+2s_2+1} g(\tau, y(\tau)) d\tau \quad (= I_{12}).
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
0 \le |I_{11}| &\le \frac{\|g\|}{\Gamma(\frac{1}{2}s_1+2s_2+2)} \int_0^{t_1} \left((t_2-\tau)^{\frac{1}{2}s_1+2s_2+1} - (t_1-\tau)^{\frac{1}{2}s_1+2s_2+1} \right) d\tau \\
&= \frac{\|g\|}{\Gamma(\frac{1}{2}s_1+2s_2+2)} \frac{t_2^{\frac{1}{2}s_1+2s_2+2} - (t_2-t_1)^{\frac{1}{2}s_1+2s_2+2} - t_1^{\frac{1}{2}s_1+2s_2+2}}{\frac{1}{2}s_1+2s_2+2} \\
&\le \frac{\|g\|}{\Gamma(\frac{1}{2}s_1+2s_2+2)} \frac{t_2^{\frac{1}{2}s_1+2s_2+2} - t_1^{\frac{1}{2}s_1+2s_2+2}}{\frac{1}{2}s_1+2s_2+2}.
\end{aligned}
$$

By the mean value theorem,

$$
\frac{t_2^{\frac{1}{2}s_1+2s_2+2} - t_1^{\frac{1}{2}s_1+2s_2+2}}{t_2 - t_1} = \left(\frac{1}{2}s_1 + 2s_2 + 2 \right) \theta^{\frac{1}{2}s_1+2s_2+1} \le \left(\frac{1}{2}s_1 + 2s_2 + 2 \right) T^{\frac{1}{2}s_1+2s_2+1},
$$

where $\theta \in (t_1, t_2)$. This further deduces that

$$
|I_{11}| \le \frac{\|g\|}{\Gamma(\frac{1}{2}s_1+2s_2+2)} (t_2 - t_1) T^{\frac{1}{2}s_1+2s_2+1}.
$$

As for I_{12}, we get

$$|I_{12}| \le \frac{\|g\|}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 2)} \int_{t_1}^{t_2} (t_2 - \tau)^{\frac{1}{2}s_1 + 2s_2 + 1} d\tau \le \frac{\|g\|}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 2)} (t_2 - t_1) T^{\frac{1}{2}s_1 + 2s_2 + 1},$$

using the fact that

$$(t_2 - \tau)^{\frac{1}{2}s_1 + 2s_2 + 1} \le T^{\frac{1}{2}s_1 + 2s_2 + 1}.$$

Thus

$$\begin{aligned} |I_1| &\le 2\|g\| T (t_2 - t_1) \sum_{s=0}^{\infty} \sum_{s_1 + s_2 = s} \binom{s}{s_1, s_2} |b|^{s_1} |c|^{s_2} \frac{T^{\frac{1}{2}s_1 + 2s_2}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 2)} \\ &= 2\|g\| T (t_2 - t_1) E_{(1/2, 2), 2}\left(|b| T^{1/2}, |c| T^2\right), \end{aligned}$$

which claims that I_1 is equicontinuous on $[0, T]$.

On the other hand, we infer from Section 1 that

$$I_2 + I_4 = O(t_2^2 - t_1^2) = O((t_2 - t_1)(t_2 + t_1)) = O(t_2 - t_1), \quad \text{since } t_2 + t_1 \le 2T,$$

using the fact that

$$\sum_{s_1 + s_2 = 0} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t_2^{\frac{1}{2}s_1 + 2s_2} - t_1^{\frac{1}{2}s_1 + 2s_2}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 1)} = 0.$$

This indicates that $I_2 + I_4$ is equicontinuous. Similarly, from

$$I_3 + I_5 = \phi_2(y)(t_2 - t_1) + O(t_2^3 - t_1^3) = \phi_2(y)(t_2 - t_1) + O((t_2 - t_1)(t_2^2 + t_2 t_1 + t_1^2))$$

is equicontinuous as $\phi_2(y)$ and $t_2^2 + t_2 t_1 + t_1^2$ are bounded. In summary, $\mathcal{M}$ is completely continuous.

(iv) Finally, we will show that for any $\beta \in (0, 1]$, the set

$$W_0 = \{y \in C[0, T] : y = \beta \mathcal{M} y\}$$

is bounded. It follows that

$$\begin{aligned} \|y\| \le \|\mathcal{M} y\| &\le \|g\| T^2 E_{(1/2, 2), 3}\left(|b| T^{1/2}, |c| T^2\right) + |\phi_1(y)| E_{(1/2, 2), 1}\left(|b| T^{1/2}, |c| T^2\right) \\ &+ |\phi_2(y)| T E_{(1/2, 2), 2}\left(|b| T^{1/2}, |c| T^2\right) + |b \phi_1(y)| T^{1/2} E_{(1/2, 2), 3/2}\left(|b| T^{1/2}, |c| T^2\right) \\ &+ |b \phi_2(y)| T^{3/2} E_{(1/2, 2), 5/2}\left(|b| T^{1/2}, |c| T^2\right). \end{aligned}$$

Using

$$\begin{aligned} &|\phi_1(y)| \le \Omega_1 \|y\| + |\phi_1(0)|, \quad \text{and} \\ &|\phi_2(y)| \le \Omega_2 \|y\| + |\phi_2(0)|, \end{aligned}$$

we arrive at

$$\begin{aligned} \|y\| &\le \|g\| T^2 E_{(1/2, 2), 3}\left(|b| T^{1/2}, |c| T^2\right) + \Omega_1 \|y\| E_{(1/2, 2), 1}\left(|b| T^{1/2}, |c| T^2\right) \\ &+ |\phi_1(0)| E_{(1/2, 2), 1}\left(|b| T^{1/2}, |c| T^2\right) + \Omega_2 \|y\| T E_{(1/2, 2), 2}\left(|b| T^{1/2}, |c| T^2\right) \\ &+ |\phi_2(0)| T E_{(1/2, 2), 2}\left(|b| T^{1/2}, |c| T^2\right) + |b| \Omega_1 T^{1/2} \|y\| E_{(1/2, 2), 3/2}\left(|b| T^{1/2}, |c| T^2\right) \\ &+ |b \phi_1(0)| T^{1/2} E_{(1/2, 2), 3/2}\left(|b| T^{1/2}, |c| T^2\right) + |b| \Omega_2 T^{3/2} \|y\| E_{(1/2, 2), 5/2}\left(|b| T^{1/2}, |c| T^2\right) \\ &+ |b \phi_2(0)| T^{3/2} E_{(1/2, 2), 5/2}\left(|b| T^{1/2}, |c| T^2\right). \end{aligned}$$

Let

$$B_0 = 1 - \Omega_1 E_{(1/2,2),1}\Big(|b|T^{1/2}, |c|T^2\Big) - \Omega_2 T E_{(1/2,2),2}\Big(|b|T^{1/2}, |c|T^2\Big)$$
$$- |b|\Omega_1 T^{1/2} E_{(1/2,2),3/2}\Big(|b|T^{1/2}, |c|T^2\Big) - |b|\Omega_2 T^{3/2} E_{(1/2,2),5/2}\Big(|b|T^{1/2}, |c|T^2\Big) > 0.$$

Then

$$\|y\| \le \frac{\|g\|T^2}{B_0} E_{(1/2,2),3}\Big(|b|T^{1/2}, |c|T^2\Big) + \frac{|\phi_1(0)|}{B_0} E_{(1/2,2),1}\Big(|b|T^{1/2}, |c|T^2\Big)$$
$$+ \frac{|\phi_2(0)|T}{B_0} E_{(1/2,2),2}\Big(|b|T^{1/2}, |c|T^2\Big) + \frac{|b\phi_1(0)|T^{1/2}}{B_0} E_{(1/2,2),3/2}\Big(|b|T^{1/2}, |c|T^2\Big)$$
$$+ \frac{|b\phi_2(0)|T^{3/2}}{B_0} E_{(1/2,2),5/2}\Big(|b|T^{1/2}, |c|T^2\Big),$$

which is bounded. By Leray–Schauder's fixed-point theorem, Equation (1) has a solution in $C[0,T]$. This completes the proof. □

Remark 2. *We should point out that if ϕ_1 and ϕ_2 are bounded functionals then the condition that $B_0 > 0$ in Theorem 2 can be removed from its proof as*

$$\|y\| \le \|\mathcal{M}y\| \le \|g\|T^2 E_{(1/2,2),3}\Big(|b|T^{1/2}, |c|T^2\Big) + |\phi_1(y)| E_{(1/2,2),1}\Big(|b|T^{1/2}, |c|T^2\Big)$$
$$+ |\phi_2(y)| T E_{(1/2,2),2}\Big(|b|T^{1/2}, |c|T^2\Big) + |b\phi_1(y)|T^{1/2} E_{(1/2,2),3/2}\Big(|b|T^{1/2}, |c|T^2\Big)$$
$$+ |b\phi_2(y)|T^{3/2} E_{(1/2,2),5/2}\Big(|b|T^{1/2}, |c|T^2\Big)$$

is bounded.

4. Hyers-Ulam's Stability

Theorem 3. *Assume g is a continuous and bounded function on $[0,T] \times \mathbb{R}$ and satisfies the Lipschitz condition for a constant $\mathcal{L} \ge 0$:*

$$|g(t,y_1) - g(t,y_2)| \le \mathcal{L}|y_1 - y_2|, \quad \textit{for all } y_1, y_2 \in \mathbb{R},$$

and ϕ_1 and ϕ_2 are functionals from $C[0,T]$ to $\mathbb{R}$ that satisfy the conditions for $\Omega_1, \ \Omega_2 \ge 0$:

$$|\phi_1(y_1) - \phi_1(y_2)| \le \Omega_1 \|y_1 - y_2\|,$$
$$|\phi_2(y_1) - \phi_2(y_2)| \le \Omega_2 \|y_1 - y_2\|,$$

where for all $y_1, \ y_2 \in C[0,T]$. In addition,

$$B_0 = 1 - \Omega_1 E_{(1/2,2),1}\Big(|b|T^{1/2}, |c|T^2\Big) - \Omega_2 T E_{(1/2,2),2}\Big(|b|T^{1/2}, |c|T^2\Big)$$
$$- |b|\Omega_1 T^{1/2} E_{(1/2,2),3/2}\Big(|b|T^{1/2}, |c|T^2\Big) - |b|\Omega_2 T^{3/2} E_{(1/2,2),5/2}\Big(|b|T^{1/2}, |c|T^2\Big) > 0,$$

and

$$\mathcal{C}_0 = 1 - \mathcal{L}T^2 E_{(1/2,2),3}\Big(|b|T^{1/2}, |c|T^2\Big) > 0.$$

Then Equation (1) is Hyers-Ulam stable.

Proof. For each $\epsilon > 0$, we assume that the function $y \in C^2[0,T]$ satisfies

$$\left\| y^{(2)}(t) + b\, {}_cD^{3/2} y(t) + c\, y(t) - g(t, y(t)) \right\| \le \epsilon.$$

Let

$$\eta_y(t) = y^{(2)}(t) + b\, {}_cD^{3/2}y(t) + c\, y(t) - g(t, y(t)).$$

Then

$$\|\eta_y\| \leq \epsilon$$

and equation

$$y^{(2)}(t) + b\, {}_cD^{3/2}y(t) + c\, y(t) = g(t, y(t)) + \eta_y(t)$$

has a solution according to Theorem 2 since the function $g(t, y(t)) + \eta_y(t)$ is bounded and continuous, and $B_0 > 0$. Therefore,

$$\begin{aligned}
y(t) = {} & \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} I^{\frac{1}{2}s_1+2s_2+2}(g(t, y(t)) + \eta_y(t)) \\
& + y(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 1)} \\
& + y'(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+1}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 2)} \\
& + b\, y(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{1}{2}}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + \frac{3}{2})} \\
& + b\, y'(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{3}{2}}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + \frac{5}{2})}.
\end{aligned}$$

As Equation (1) also has a solution $y_0(t)$ by Theorem 2 with the initial conditions $y_0(0) = y(0)$ and $y_0'(0) = y'(0)$ as $B_0 > 0$, we come to

$$\begin{aligned}
y_0(t) = {} & \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} I^{\frac{1}{2}s_1+2s_2+2} g(t, y_0(t)) \\
& + y(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 1)} \\
& + y'(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+1}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + 2)} \\
& + b\, y(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{1}{2}}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + \frac{3}{2})} \\
& + b\, y'(0) \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} \frac{t^{\frac{1}{2}s_1+2s_2+\frac{3}{2}}}{\Gamma(\frac{1}{2}s_1 + 2s_2 + \frac{5}{2})}.
\end{aligned}$$

Then

$$\begin{aligned}
y(t) - y_0(t) = {} & \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} I^{\frac{1}{2}s_1+2s_2+2}(g(t, y(t)) - g(t, y_0(t))) \\
& + \sum_{s=0}^{\infty} \sum_{s_1+s_2=s} (-1)^s \binom{s}{s_1, s_2} b^{s_1} c^{s_2} I^{\frac{1}{2}s_1+2s_2+2} \eta_y(t).
\end{aligned}$$

Hence,

$$\|y - y_0\| \leq \mathcal{L}\|y - y_0\| T^2 E_{(1/2,2),\,3}\Big(|b|T^{1/2},\ |c|T^2\Big) + T^2 E_{(1/2,2),\,3}\Big(|b|T^{1/2},\ |c|T^2\Big)\epsilon.$$

Choosing

$$\mathcal{C}_0 = 1 - \mathcal{L} T^2 E_{(1/2,2),\,3}\Big(|b|T^{1/2},\ |c|T^2\Big) > 0.$$

This implies that
$$\|y - y_0\| \le K\epsilon,$$
where
$$K = \frac{1}{\mathcal{C}_0} T^2 E_{(1/2,2),\,3}\Big(|b|T^{1/2},\ |c|T^2\Big),$$
is independent of ϵ, y_0 and y. This completes the proof. □

Remark 3. *Owing to the compactness of the interval $[0, T]$ and the continuity of both the exact and approximate solutions, unbounded deviation between them cannot occur. Consequently, the Hyers–Ulam stability of Equation (1) on $[0, T]$ is anticipated.*

5. Illustrative Examples

Example 1. *The following nonlinear Bagley–Torvik equation:*
$$\begin{cases} y^{(2)}(t) - \dfrac{1}{9}\,{}_cD^{3/2}y(t) - \dfrac{1}{5}\,y(t) = \dfrac{1}{20}\sin\left(\dfrac{y(t)}{t^2+1}\right) + \dfrac{t}{t^2+3}, & t \in [0, \sqrt{\pi}\,], \\ y(0) = \dfrac{1}{29}\displaystyle\int_0^{\sqrt{\pi}} t|y(t)|dt, \quad y'(0) = \dfrac{1}{18}\cos y(3/4), \end{cases} \tag{7}$$
has a unique solution in $C[0, \sqrt{\pi}\,]$

Proof. Let
$$g(t, y) = \frac{1}{20}\sin\left(\frac{y}{t^2+1}\right) + \frac{t}{t^2+3}.$$
Then g is a continuous and bounded function on $C[0, \sqrt{\pi}\,] \times \mathbb{R}$, satisfying the Lipschitz condition:
$$|g(t, \zeta_1) - g(t, \zeta_2)| \le \frac{1}{20}\left|\frac{\zeta_1}{t^2+1} - \frac{\zeta_2}{t^2+1}\right| \le \frac{1}{20}|\zeta_1 - \zeta_2|,$$
which implies that $\Omega_0 = \dfrac{1}{20}$. On the other hand,
$$\phi_1(y) = \frac{1}{29}\int_0^{\sqrt{\pi}} t|y(t)|dt$$
is a functional satisfying
$$\begin{aligned} |\phi_1(y_1) - \phi_1(y_2)| &\le \frac{1}{29}\int_0^{\sqrt{\pi}} |t(|y_1(t)| - |y_2(t)|)|dt \\ &\le \frac{\sqrt{\pi}}{29}\int_0^{\sqrt{\pi}} |y_1(t) - y_2(t)|dt \\ &\le \frac{\pi}{29}\|y_1 - y_2\|, \end{aligned}$$
which implies that $\Omega_1 = \dfrac{\pi}{29}$. Similarly,
$$\phi_2(y) = \frac{1}{18}\cos y(3/4)$$
satisfies
$$\begin{aligned} |\phi_2(y_1) - \phi_2(y_2)| &\le \frac{1}{18}|\cos y_1(3/4) - \cos y_2(3/4)| \\ &\le \frac{1}{18}|y_1(3/4) - y_1(3/4)| \\ &\le \frac{1}{18}\|y_1 - y_2\|, \end{aligned}$$

which indicates that $\Omega_2 = \dfrac{1}{18}$. We need to evaluate the value of

$$\begin{aligned}
B &= \Omega_0 T^2 E_{(1/2,2),\,3}\left(|b|T^{1/2},\,|c|T^2\right) + \Omega_1 E_{(1/2,2),\,1}\left(|b|T^{1/2},\,|c|T^2\right) \\
&\quad + \Omega_2 T E_{(1/2,2),\,2}\left(|b|T^{1/2},\,|c|T^2\right) + |b|\Omega_1 T^{1/2} E_{(1/2,2),\,3/2}\left(|b|T^{1/2},\,|c|T^2\right) \\
&\quad + |b|\Omega_2 T^{3/2} E_{(1/2,2),\,5/2}\left(|b|T^{1/2},\,|c|T^2\right) \\
&= \frac{(\sqrt{\pi})^2}{20} E_{(1/2,2),\,3}\left(\left|-\frac{1}{9}\right|(\sqrt{\pi})^{1/2},\,\left|-\frac{1}{5}\right|(\sqrt{\pi})^2\right) \\
&\quad + \frac{\pi}{29} E_{(1/2,2),\,1}\left(\left|-\frac{1}{9}\right|(\sqrt{\pi})^{1/2},\,\left|-\frac{1}{5}\right|(\sqrt{\pi})^2\right) \\
&\quad + \frac{\sqrt{\pi}}{18} E_{(1/2,2),\,2}\left(\left|-\frac{1}{9}\right|(\sqrt{\pi})^{1/2},\,\left|-\frac{1}{5}\right|(\sqrt{\pi})^2\right) \\
&\quad + \left|-\frac{1}{9}\right|\left(\frac{\pi}{29}\right)(\sqrt{\pi})^{1/2} E_{(1/2,2),\,3/2}\left(\left|-\frac{1}{9}\right|(\sqrt{\pi})^{1/2},\,\left|-\frac{1}{5}\right|(\sqrt{\pi})^2\right) \\
&\quad + \left|-\frac{1}{9}\right|\left(\frac{(\sqrt{\pi})^{3/2}}{18}\right) E_{(1/2,2),\,5/2}\left(\left|-\frac{1}{9}\right|(\sqrt{\pi})^{1/2},\,\left|-\frac{1}{5}\right|(\sqrt{\pi})^2\right).
\end{aligned}$$

By our Python (latest v. 3.15) codes, we have computed that

$$\begin{aligned}
B &\approx \frac{(\sqrt{\pi})^2}{20} * 0.5792242101936123 + \frac{\pi}{29} * 1.5899492944462168 \\
&\quad + \frac{\sqrt{\pi}}{18} * 1.2499074435613728 + \left|-\frac{1}{9}\right|\left(\frac{\pi}{29}\right)(\sqrt{\pi})^{1/2} * 1.5278704096489806 \\
&\quad + \left|-\frac{1}{9}\right|\left(\frac{(\sqrt{\pi})^{3/2}}{18}\right) * 0.8976756036026237 \\
&\approx 0.4238625457606517 < 1.
\end{aligned}$$

Therefore, Equation (7) has a unique solution from Theorem 1. This completes the proof. □

Example 2. *The following nonlinear Bagley–Torvik equation:*

$$\begin{cases} y^{(2)}(t) + \dfrac{1}{10}\,{}_cD^{3/2}y(t) - \dfrac{1}{8}\,y(t) = \dfrac{\sqrt{|y(t)|}}{1+y^2(t)} + \dfrac{\arctan(t)}{2}, \quad t \in [0,2], \\ y(0) = \dfrac{1}{16}\sin(y(1)+1), \quad y'(0) = \dfrac{1}{7}\arctan y(1/2), \end{cases} \tag{8}$$

has at least one solution in $C[0,2]$.

Clearly,

$$g(t,y) = \frac{\sqrt{|y|}}{1+y^2} + \frac{arctan(t)}{2}$$

is a continuous and bounded function on $[0,2] \times \mathbb{R}$ and

$$\phi_1(y) = \frac{1}{16}\sin(y(1)+1)$$

satisfies

$$|\phi_1(y_1) - \phi_1(y_2)| \le \frac{1}{16}|\sin(y_1(1)+1) - \sin(y_2(1)+1)| \le \frac{1}{16}\|y_1 - y_2\|.$$

Thus, $\Omega_1 = \frac{1}{16}$. Similarly,

$$\phi_2(y) = \frac{1}{7} \arctan y(1/2)$$

satisfies

$$\begin{aligned}|\phi_2(y_1) - \phi_2(y_2)| &\le \frac{1}{7} |\arctan y_1(1/2) - \arctan y_2(1/2)| \\ &\le \frac{1}{7} |y_1(1/2) - y_2(1/2)| \\ &\le \frac{1}{7} \|y_1 - y_2\|,\end{aligned}$$

which claims that $\Omega_2 = \frac{1}{7}$. We need to compute

$$\begin{aligned}B_0 &= 1 - \Omega_1 E_{(1/2,2),\,1}\left(|b|T^{1/2},\, |c|T^2\right) - \Omega_2 T E_{(1/2,2),\,2}\left(|b|T^{1/2},\, |c|T^2\right) \\ &\quad - |b|\Omega_1 T^{1/2} E_{(1/2,2),\,3/2}\left(|b|T^{1/2},\, |c|T^2\right) - |b|\Omega_2 T^{3/2} E_{(1/2,2),\,5/2}\left(|b|T^{1/2},\, |c|T^2\right) \\ &= 1 - \frac{1}{16} E_{(1/2,2),\,1}\left(\left|\frac{1}{10}\right| 2^{1/2},\, \left|-\frac{1}{8}\right| 2^2\right) - \frac{2}{7} E_{(1/2,2),\,2}\left(\left|\frac{1}{10}\right| 2^{1/2},\, \left|-\frac{1}{8}\right| 2^2\right) \\ &\quad - \left|\frac{1}{10}\right| \left(\frac{2^{1/2}}{16}\right) E_{(1/2,2),\,3/2}\left(\left|\frac{1}{10}\right| 2^{1/2},\, \left|-\frac{1}{8}\right| 2^2\right) \\ &\quad - \left|\frac{1}{10}\right| \left(\frac{2^{3/2}}{7}\right) E_{(1/2,2),\,5/2}\left(\left|\frac{1}{10}\right| 2^{1/2},\, \left|-\frac{1}{8}\right| 2^2\right).\end{aligned}$$

Using our Python codes, we have calculated that

$$\begin{aligned}B_0 &\approx 1 - \frac{1}{16} * 1.4929471511906593 - \frac{2}{7} * 1.2166944561361335 \\ &\quad - \left|\frac{1}{10}\right| \left(\frac{2^{1/2}}{16}\right) * 1.469105034028516 - \left|\frac{1}{10}\right| \left(\frac{2^{3/2}}{7}\right) * 0.8800364573467343 \\ &\approx 0.5105197962676558 > 0.\end{aligned}$$

By Theorem 2, Equation (8) has at least one solution in $C[0,2]$. The proof is completed.

Remark 4. *Indeed, from Remark 2 we can claim that Equation (8) has at least one solution in* $C[0,2]$ *without computing* B_0 *in Theorem 2 since* ϕ_1 *and* ϕ_2 *are bounded.*

Example 3. *The following nonlinear Bagley–Torvik equation:*

$$\begin{cases} y^{(2)}(t) + \frac{1}{20}\, {}_cD^{3/2} y(t) + \frac{1}{6}\, y(t) = \frac{1}{30(1 + y^2(t))} + t^3 + 1, \quad t \in [0,3], \\ y(0) = \frac{1}{15(|y(2)| + 1)}, \quad y'(0) = \frac{1}{28} \int_0^3 |y(t)|dt, \end{cases} \tag{9}$$

is Hyers-Ulam stable.

Proof. Since

$$g(t,y) = \frac{1}{30(1+y^2)} + t^3 + 1$$

is a continuous and bounded function on $[0,3] \times \mathbb{R}$, satisfying

$$|g(t,\zeta_1) - g(t,\zeta_2)| \le \frac{1}{30} \left| \frac{1}{1+\zeta_1^2} - \frac{1}{1+\zeta_2^2} \right| \le \frac{1}{30} |\zeta_1 - \zeta_2|, \quad \text{for } \zeta_1, \zeta_2 \in \mathbb{R},$$

which infers that $\mathcal{L} = \dfrac{1}{30}$. On the other hand,

$$\phi_1(y) = \frac{1}{15(|y(2)| + 1)}$$

is a functional satisfying

$$|\phi_1(y_1) - \phi_1(y_2)| \leq \frac{1}{15}\left|\frac{1}{|y_1(2)| + 1} - \frac{1}{|y_2(2)| + 1}\right| \leq \frac{1}{15}\|y_1 - y_2\|,$$

which claims that $\Omega_1 = \dfrac{1}{15}$. Similarly,

$$\phi_2(y) = \frac{1}{28}\int_0^3 |y(t)|dt$$

satisfies

$$|\phi_2(y_1) - \phi_2(y_2)| \leq \frac{1}{28}\int_0^3 |(|y_1(t)| - |y_2(t)|)|dt \leq \frac{3}{28}\|y_1 - y_2\|,$$

which implies that $\Omega_2 = \dfrac{3}{28}$. According to Theorem 3, we need to evaluate the value of

$$\begin{aligned}
B_0 &= 1 - \Omega_1 E_{(1/2,2),\,1}\Big(|b|T^{1/2},\, |c|T^2\Big) - \Omega_2 T E_{(1/2,2),\,2}\Big(|b|T^{1/2},\, |c|T^2\Big) \\
&\quad - |b|\Omega_1 T^{1/2} E_{(1/2,2),\,3/2}\Big(|b|T^{1/2},\, |c|T^2\Big) - |b|\Omega_2 T^{3/2} E_{(1/2,2),\,5/2}\Big(|b|T^{1/2},\, |c|T^2\Big) \\
&= 1 - \frac{1}{15}E_{(1/2,2),\,1}\left(\left|\frac{1}{20}\right|3^{1/2},\, \left|\frac{1}{6}\right|3^2\right) - \frac{9}{28}E_{(1/2,2),\,2}\left(\left|\frac{1}{20}\right|3^{1/2},\, \left|\frac{1}{6}\right|3^2\right) \\
&\quad - \left|\frac{1}{20}\right|\left(\frac{3^{1/2}}{15}\right)E_{(1/2,2),\,3/2}\left(\left|\frac{1}{20}\right|3^{1/2},\, \left|\frac{1}{6}\right|3^2\right) \\
&\quad - \left|\frac{1}{20}\right|\left(\frac{3}{28}\right)3^{3/2}E_{(1/2,2),\,5/2}\left(\left|\frac{1}{20}\right|3^{1/2},\, \left|\frac{1}{6}\right|3^2\right).
\end{aligned}$$

Using our Python codes, we have computed that

$$\begin{aligned}
B_0 &\approx 1 - \frac{1}{15} * 2.05122290580332 - \frac{9}{28} * 1.3646280206071075 \\
&\quad - \left|\frac{1}{20}\right|\left(\frac{3^{1/2}}{15}\right) * 1.769026295739645 - \left|\frac{1}{20}\right|\left(\frac{3}{28}\right)3^{3/2} * 0.9473474103320245 \\
&\approx 0.38803702760410463 > 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
C_0 &= 1 - \mathcal{L}T^2 E_{(1/2,2),\,3}\Big(|b|T^{1/2},\, |c|T^2\Big) = 1 - \frac{9}{30}E_{(1/2,2),\,3}\left(\frac{1}{20}3^{1/2}, \frac{1}{6}3^2\right) \\
&\approx 0.820396158205229 > 0.
\end{aligned}$$

Hence, Equation (9) is Hyers-Ulam stable in $C[0,3]$ by Theorem 3. This completes the proof. □

6. Conclusions

By applying the inverse operator method along with several well-known fixed-point theorems, we have investigated the existence, uniqueness, and Hyers–Ulam stability of

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solutions to Equation (1), utilizing the multivariate Mittag-Leffler function. Several illustrative examples have been provided to demonstrate the applicability of the main results.

As a direction for future research, it would be of interest to consider the following nonlinear Bagley–Torvik equation with functional boundary conditions by using the inverse operator method and fixed-point theorems:

$$\begin{cases} y^{(2)}(t) + b\,{}_cD^{3/2}y(t) + c\,y(t) = g(t, y(t)), \quad t \in [0, T], \\ y(0) = \phi_1(y), \quad y(T) = \phi_2(y), \end{cases}$$

as well as the equation with a variable coefficient

$$\begin{cases} y^{(2)}(t) + b(t)\,{}_cD^{3/2}y(t) + c(t)\,y(t) = g(t, y(t)), \quad t \in [0, T], \\ y(0) = \phi_1(y), \quad y(T) = \phi_2(y). \end{cases}$$

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