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Article

A Generalized Nonlinear Bagley–Torvik Equation in Distributions

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Abstract

This paper investigates the fractional calculus of distributions supported on $\mathbb{R}^+$ in the sense of L. Schwartz, based on distributional convolutions. We further study a generalized Bagley–Torvik equation involving an arbitrary number of fractional derivative terms with orders in the interval $(0,2)$. The existence and uniqueness of solutions for its nonlinear form are established in a space of continuous functions by applying Banach's contraction principle, the Leray–Schauder fixed-point theorem, inverse operators, and the multivariate Mittag–Leffler function. Finally, several examples are presented, in which the values of multivariate Mittag–Leffler functions are computed to illustrate the main results.

Keywords: generalized nonlinear Bagley–Torvik equation; fixed-point theorem; multivariate Mittag–Leffler function; inverse operator; distribution

MSC: 46F10; 34A12; 26A33

1. Introduction

In this section, we study the fundamental concepts of distributions and convolutions in the sense of Schwartz, explore the fractional derivatives and integrals of distributions, discuss the method of inverse operators, and present applications of fractional differential equations formulated in the distributional framework.

1.1. Distributions

To study fractional calculus of certain types of distributions, we begin introducing the following definitions in detail, which can be found in [1,2]. Let $\mathcal{D}(\mathbb{R})$ be the space of infinitely differentiable functions with compact support in $\mathbb{R}$, and $\mathcal{D}'(\mathbb{R})$ be the space of linear and continuous functionals (distributions) defined on $\mathcal{D}(\mathbb{R})$. Further, we define a sequence $\{\phi_n(x)\} \subset \mathcal{D}(\mathbb{R})$ converging to zero if all these functions vanish outside a fixed bounded interval, and converge uniformly to zero in the usual sense together with their derivatives of any order. Clearly, the functional δ given by

$$(\delta, \phi) = \phi(0)$$

is a distribution in $\mathcal{D}'(\mathbb{R})$, as it is linear and continuous on $\mathcal{D}(\mathbb{R})$. Let f be a locally integrable function on $\mathbb{R}$. We define

$$(f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx,$$

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which is a distribution in $\mathcal{D}'(\mathbb{R})$ by noting that the support of ϕ is bounded and the integral clearly exists.

Let $f \in \mathcal{D}'(\mathbb{R})$. The distributional derivative of f, denoted by f' or df/dx, is defined as

$$(f',\ \phi) = \left(\frac{df}{dx},\ \phi\right) = -(f, \phi'),$$

where $\phi \in \mathcal{D}(\mathbb{R})$.

Clearly, $f' \in \mathcal{D}'(\mathbb{R})$ and every distribution has a derivative. We define the Heaviside function $\theta(x)$ as

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases}$$

which is undefined at $x = 0$ (hence, it is discontinuous). But, the integral

$$(\theta, \phi) = \int_0^\infty \phi(x)dx \quad \text{for} \quad \phi \in \mathcal{D}(\mathbb{R})$$

is a distribution in $\mathcal{D}'(\mathbb{R})$. As an example, we are going to prove that $\theta' = \delta$. Indeed,

$$(\theta',\ \phi) = -(\theta,\ \phi') = -\int_0^\infty \phi' dx = \phi(0) = (\delta,\ \phi),$$

which claims that $\theta' = \delta$. Typically, a distribution does not have a well-defined value at a point, such as $\delta(0)$.

Furthermore, within the framework of Schwartz distribution theory, it is generally impossible to define the product of two arbitrary distributions [3]. However, the product of an infinitely differentiable function $\psi(x)$ with a distribution f is given by

$$(\psi f,\ \phi) = (f, \psi\phi)$$

which is well-defined since $\psi\phi \in \mathcal{D}(\mathbb{R})$ if $\phi \in \mathcal{D}(\mathbb{R})$.

Let ψ be an infinitely differentiable function. Then, the product of $\psi(x)\delta^{(n)}(x)$ exists for all $n = 0, 1, \cdots$, and

$$\psi(x)\delta^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \psi^{(k)}(0)\delta^{(n-k)}(x).$$

In particular,

$$x\delta = 0, \quad x\delta' = -\delta,$$
$$x^n\delta^n = (-1)^n n!\delta.$$

We now consider the distribution x_+^λ given by

$$x_+^\lambda = \begin{cases} x^\lambda, & \text{if } x > 0, \\ 0, & \text{if } x \le 0, \end{cases}$$

where λ is a complex number. This distribution will play an important role in defining the fractional derivatives and integrals of distributions in $\mathcal{D}'(\mathbb{R}^+)$, which is a subspace of $\mathcal{D}'(\mathbb{R})$. Obviously, the integral defined by x_+^λ for $\mathrm{Re}\lambda > -1$,

$$(x_+^\lambda, \phi) = \int_0^\infty x^\lambda \phi(x)dx,$$

is regular, which can be analytically continued to $\text{Re}\lambda > -2, \quad \lambda \neq -1$ by the identity

$$\int_0^\infty x^\lambda \phi(x) dx = \int_0^1 x^\lambda [\phi(x) - \phi(0)] dx + \int_1^\infty x^\lambda \phi(x) dx + \frac{\phi(0)}{\lambda + 1}.$$

This is well-defined for $\text{Re}\lambda > -1$. In particular, for $\text{Re}\lambda > -2, \quad \lambda \neq -1$, the right-hand side exists and defines a normalization of the integral on the left.

We can similarly extend x_+^λ to the region $\text{Re}\lambda > -n-1, \quad \lambda \neq -1, -2, \cdots, -n$ to get

$$\begin{aligned} \int_0^\infty x^\lambda \phi(x) dx = &\int_0^1 x^\lambda \left[\phi(x) - \phi(0) - \cdots - \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0) \right] dx \\ &+ \int_1^\infty x^\lambda \phi(x) dx + \sum_{k=1}^{n} \frac{\phi^{(k-1)}(0)}{(k-1)!(\lambda + k)}. \end{aligned} \quad (1)$$

Clearly, the right-hand side regularizes the integral on the left. This defines the distribution x_+^λ for $\text{Re}\lambda > -n-1, \quad \lambda \neq -1, -2, \cdots, -n$. Furthermore, if $-n-1 < \text{Re}\lambda < -n$, we derive that

$$(x_+^\lambda, \phi) = \int_0^\infty x^\lambda \left[\phi(x) - \phi(0) - \cdots - \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0) \right] dx, \quad (2)$$

by noting that ϕ has bounded support.

In addition, Equation (1) shows that when we treat (x_+^λ, ϕ) as a function of λ, it has simple poles at $\lambda = -1, -2, \cdots$, with its residue at $\lambda = -k$ being

$$\frac{\phi^{(k-1)}(0)}{(k-1)!} = \frac{(-1)^{k-1}}{(k-1)!} (\delta^{(k-1)}(x), \phi(x)).$$

Hence, we imply that the functional x_+^λ has a simple pole at $\lambda = -k$, and the residue there is

$$\frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(x).$$

For $\text{Re}\lambda > 0$, we come to

$$(x_+^\lambda, \phi'(x)) = \int_0^\infty x^\lambda \phi'(x) dx = -(\lambda x_+^{\lambda-1}. \phi(x)).$$

Since both sides of the above equation can be analytically continued to the entire plane except $\lambda = -1, -2, \cdots$, the uniqueness of analytic continuation implies that the following equation holds in $\mathbb{C}$:

$$\frac{dx_+^\lambda}{dx} = \lambda x_+^{\lambda-1}, \quad \lambda \neq -1, -2, \cdots.$$

The Gamma function is defined as

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} e^{-x} dx,$$

which converges for $\text{Re}\lambda > 0$. This integral can be considered as the application of $x_+^{\lambda-1}$ to the test function e^{-x} on $\mathbb{R}^+$. For $\text{Re}\lambda > -n-1, \quad \lambda \neq -1, -2, \cdots, -n$, we get the following by using Equation (1):

$$\Gamma(\lambda) = \int_0^1 x^{\lambda-1} \left[e^{-x} - \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} \right] dx + \int_1^\infty x^{\lambda-1} e^{-x} + \sum_{k=0}^{n} \frac{(-1)^k}{k!(\lambda + n)}.$$

For $-n-1 < \text{Re}\lambda < -n$, we deduce the following by Equation (2):

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1}\left[e^{-x} - \sum_{k=0}^{n}(-1)^k \frac{x^k}{k!}\right]dx.$$

We further claim that

$$\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}(\mathbb{R}^+)$$

is an entire function of λ on the complex plane $\mathbb{C}$. In fact,

$$\left.\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right|_{\lambda=-n} = \frac{\text{res}_{\lambda=-n}x_+^{\lambda-1}}{\text{res}_{\lambda=-n}(x_+^{\lambda-1},\, e^{-x})} = \frac{(-1)^n\delta^{(n)}(x)n!}{(-1)^n(\delta^{(n)}(x),\, e^{-x})n!} = \delta^{(n)}(x). \tag{3}$$

Moreover, the derivative of Φ_λ is simpler than that for x_+^λ. Indeed,

$$\frac{d}{dx}\Phi_\lambda = \frac{d}{dx}\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{(\lambda-1)x_+^{\lambda-2}}{\Gamma(\lambda)} = \frac{x_+^{\lambda-2}}{\Gamma(\lambda-1)} = \Phi_{\lambda-1}. \tag{4}$$

1.2. The Convolutions of Distributions

The convolution of certain pairs of distributions is usually defined as follows [1].

Definition 1. *Let f and g be distributions in $\mathcal{D}'(\mathbb{R})$ satisfying either of the following conditions: (a) Either f or g has bounded support. (b) The supports of f and g are bounded on the same side. Then, the convolution $f * g$ is defined by the identity*

$$((f*g)(x),\, \phi(x)) = (g(x), (f(y),\, \phi(x+y)))$$

for all $\phi \in \mathcal{D}(\mathbb{R})$.

The classical definition of convolution is given below.

Definition 2. *If f and g are locally integrable functions, then the convolution $f * g$ is defined by*

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

if the integral exists.

We would like to point out that if f and g are locally integrable functions satisfying either of the conditions (a) or (b) in Definition 1, then Definition 1 is in agreement with Definition 2. Furthermore, if the convolution $f * g$ exists by Definitions 1 or 2, then the following identities hold:

$$f * g = g * f, \tag{5}$$

$$(f*g)' = f*g' = f'*g, \tag{6}$$

where all the derivatives are in the distributional sense.

Let λ and μ be arbitrary complex numbers with $\text{Re}\lambda > 0$ and $\text{Re}\mu > 0$. Then,

$$\begin{aligned}\Phi_\lambda * \Phi_\mu &= \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \frac{x_+^{\mu-1}}{\Gamma(\mu)} = \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_{-\infty}^{\infty}\zeta_+^{\lambda-1}(x-\zeta)_+^{\mu-1}d\zeta \\ &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_0^x \zeta^{\lambda-1}(x-\zeta)^{\mu-1}d\zeta = \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}x_+^{\lambda+\mu-1} = \Phi_{\lambda+\mu},\end{aligned} \tag{7}$$

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by making $\zeta = xt$ and using

$$B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}.$$

Equation (7) can now to proven for other values of $\lambda,\ \mu$ by analytic continuation, or Equation (4). For example, if $-1 < \mathrm{Re}\lambda < 0$, then

$$\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{d}{dx}\frac{x_+^{\lambda+1-1}}{\Gamma(\lambda+1)},$$

and

$$\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \frac{x_+^{\mu-1}}{\Gamma(\mu)} = \frac{d}{dx}\left(\frac{x_+^{\lambda+1-1}}{\Gamma(\lambda+1)} * \frac{x_+^{\mu-1}}{\Gamma(\mu)}\right) = \frac{d}{dx}\frac{x_+^{\lambda+\mu}}{\Gamma(\lambda+\mu+1)} = \frac{x_+^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)}.$$

1.3. The Fractional Derivatives and Integrals of Distributions

The Cauchy formula

$$g_n(x) = \int_0^x \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_2}\int_0^{\zeta_1} g(\zeta)d\zeta d\zeta_1 \cdots d\zeta_{n-1} = \frac{1}{(n-1)!}\int_0^x g(\zeta)(x-\zeta)^{n-1}d\zeta$$

changes the computation of the n-fold primitive of a function $g(x)$ defined on $\mathbb{R}^+$ to a single integral. Obviously, this formula can be written in the form

$$g_n(x) = g(x) * \frac{x_+^{n-1}}{(n-1)!} = g(x) * \frac{x_+^{n-1}}{\Gamma(n)},$$

where $g(x) = 0$ for $x < 0$.

We would like to extend this formula to the case of arbitrary complex number λ and the distributions in $\mathcal{D}'(\mathbb{R}^+)$. Thus, we define the primitive of order λ of g as the convolution in the distributional sense:

$$g_\lambda(x) = g(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = g(x) * \Phi_\lambda, \tag{8}$$

which is well-defined as the supports of g and Φ_λ are bounded on the same side.

Clearly, we have the following from Equation (3):

$$\begin{aligned}
&g_0(x) = g(x) * \Phi_0 = g(x) * \delta(x) = g(x),\\
&g_{-1}(x) = g(x) * \Phi_{-1} = g(x) * \delta'(x) = g'(x),\\
&g_{-2}(x) = g(x) * \Phi_{-2} = g(x) * \delta''(x) = g''(x),\\
&.\\
&g_1(x) = g(x) * \Phi_1 = g(x) * \theta(x) = \int_0^x g(\zeta)d\zeta,\\
&g_2(x) = g(x) * \Phi_2 = g(x) * x_+ = \int_0^x (x-\zeta)g(\zeta)d\zeta,\\
&\cdots.
\end{aligned}$$

Therefore, Equation (8) with various λ will give not only the derivatives but also the integrals of $g \in \mathcal{D}'(\mathbb{R}^+)$. We define the convolution

$$g_{-\lambda} = g(x) * \Phi_{-\lambda} = \frac{d^\lambda}{dx^\lambda}g$$

as the fractional derivative of the distribution g with order λ if $\text{Re}\lambda \geq 0$, and the fractional integral if $\text{Re}\lambda < 0$.

Let $m-1 < \lambda < m \in \mathbb{N}$ and g be a distribution in $\mathcal{D}'(\mathbb{R}^+)$. Then, from Equation (6),

$$\begin{aligned} g_{-\lambda}(x) &= g(x) * \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = g(x) * \frac{d^m}{dx^m}\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \\ &= \frac{d^m}{dx^m}\left(g(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\right) = g^{(m)}(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}, \end{aligned}$$

which claims that there is no difference between the Riemann–Liouville derivative and the Caputo derivative in the distributional sense [4–6].

From Equation (7), we have

$$\frac{d^\lambda}{dx^\lambda}\frac{x_+^\mu}{\Gamma(\mu+1)} = \frac{x_+^{\mu-\lambda}}{\Gamma(\mu+1-\lambda)}.$$

In particular, for $\mu = 0$, we obtain

$$\frac{d^\lambda}{dx^\lambda}\theta(x) = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} = \Phi_{1-\lambda}.$$

From Equation (7), we derive

$$(g * \Phi_\lambda) * \Phi_\mu = g * (\Phi_\lambda * \Phi_\mu) = g * \Phi_{\lambda+\mu},$$

where $g \in \mathcal{D}'(\mathbb{R}^+)$.

Setting $\mu = -\lambda$, we see that differentiation and integration of the same order are inverse to each other, and the sequential fractional derivative law holds by using Equation (7):

$$\frac{d^\lambda}{dx^\lambda}\left(\frac{d^\mu g}{dx^\mu}\right) = \frac{d^{\lambda+\mu} g}{dx^{\lambda+\mu}} = \frac{d^\mu}{dx^\mu}\left(\frac{d^\lambda g}{dx^\lambda}\right),$$

for any complex numbers λ and μ.

1.4. The Inverse Operator Method

Let $\alpha_1, \cdots, \alpha_m, \beta > 0$ and $z_1, \cdots, z_m \in \mathbb{C}$. Then,

$$\begin{aligned} &E_{(\alpha_1,\cdots,\alpha_m),\,\beta}(z_1,\cdots,z_m) \\ &\quad= \sum_{s=0}^{\infty}\sum_{s_1+\cdots+s_m=s}\binom{s}{s_1,\cdots,s_m}\frac{z_1^{s_1}\cdots z_m^{s_m}}{\Gamma(\alpha_1 s_1+\cdots+\alpha_m s_m+\beta)} \\ &\quad= \sum_{s=0}^{\infty}\sum_{s_1+\cdots+s_m=s}\frac{s!}{s_1!\cdots s_m!}\frac{z_1^{s_1}\cdots z_m^{s_m}}{\Gamma(\alpha_1 s_1+\cdots+\alpha_m s_m+\beta)} \end{aligned}$$

is the well-known multivariate Mittag–Leffler function [4,7], which is an entire function on the complex plane $\mathbb{C}^m$. When $m = 1$, it reduces to the following two-parameter Mittag–Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{s=0}^{\infty}\frac{z^s}{\Gamma(\alpha s+\beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C}.$$

If $\beta = 1$, we obtain the classical Mittag–Leffler function defined by

$$E_\alpha(z) = \sum_{s=0}^{\infty}\frac{z^s}{\Gamma(\alpha s+1)}, \quad \alpha > 0,\ z \in \mathbb{C}.$$

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We consider the following fractional differential equation:

$$\frac{d^{0.5}}{dx^{0.5}}u(x) + 2u(x) = \delta(x) \tag{9}$$

in $\mathcal{D}'(\mathbb{R}^+)$ to show applications of inverse operators for solving fractional differential equations in the distributional sense.

Applying $\Phi_{0.5}$ to both sides of Equation (9), we get

$$u(x) + 2\Phi_{0.5} * u(x) = \Phi_{0.5},$$

by noting that δ is an identity operator for convolution, and

$$\frac{d^{0.5}}{dx^{0.5}}u(x) = \Phi_{-0.5} * u(x).$$

Hence,

$$(\delta + 2\Phi_{0.5}) * u = \Phi_{0.5}.$$

We define an operator P as

$$P = \sum_{k=0}^{\infty} (-1)^k 2^k \Phi_{0.5k}.$$

Then, P is a mapping from $C[0,T]$ (the set of all continuous functions over $[0,T]$ with the usual norm $\|y\| = \max_{x\in[0,T]} |y(x)|$) to itself with any $T > 0$ in terms of convolution. Indeed, for $\beta \in C[0,T]$, we have

$$\|P * \beta\| \leq \sum_{k=0}^{\infty} 2^k \|\Phi_{0.5k} * \beta\| = \sum_{k=0}^{\infty} 2^k \left\| \frac{1}{\Gamma(0.5k)} \int_0^x (x-t)^{0.5k-1}\beta(t)dt \right\|$$
$$\leq \|\beta\| \sum_{k=0}^{\infty} 2^k \frac{T^{0.5k}}{\Gamma(0.5k+1)} = \|\beta\| E_{0.5,1}(2T^{0.5}) < +\infty.$$

Since $T > 0$ is arbitrary, $P * \beta$ is continuous over $[0,\infty)$ due to the above uniform convergence. In addition,

$$P * (\delta + 2\Phi_{0.5}) = (\delta + 2\Phi_{0.5}) * P = \delta.$$

Indeed,

$$P * (\delta + 2\Phi_{0.5}) = P + \sum_{k=0}^{\infty} (-1)^k 2^{k+1} \Phi_{0.5(k+1)}$$
$$= \delta + \sum_{k=1}^{\infty} (-1)^k 2^k \Phi_{0.5k} + \sum_{k=0}^{\infty} (-1)^k 2^{k+1} \Phi_{0.5(k+1)}$$
$$= \delta - \sum_{k=0}^{\infty} (-1)^k 2^{k+1} \Phi_{0.5(k+1)} + \sum_{k=0}^{\infty} (-1)^k 2^{k+1} \Phi_{0.5(k+1)} = \delta.$$

Similarly,

$$(\delta + 2\Phi_{0.5}) * P = \delta,$$

and P is a unique inverse operator of $\delta + 2\Phi_{0.5}$. Hence,

$$\begin{aligned} u(x) &= P * \Phi_{0.5} = \Phi_{0.5} + \sum_{k=1}^{\infty} (-1)^k 2^k \Phi_{0.5k} * \Phi_{0.5} \\ &= \frac{x_+^{-0.5}}{\Gamma(0.5)} + \sum_{k=1}^{\infty} (-1)^k 2^k \frac{x_+^{0.5k-0.5}}{\Gamma(0.5k+0.5)} = \frac{x_+^{-0.5}}{\sqrt{\pi}} - 2\sum_{k=0}^{\infty} (-1)^k 2^k \frac{x_+^{0.5k}}{\Gamma(0.5k+1)} \\ &= \frac{x_+^{-0.5}}{\sqrt{\pi}} - 2E_{0.5,1}(-2x_+^{0.5}). \end{aligned}$$

Clearly, $\frac{x_+^{-0.5}}{\sqrt{\pi}}$ is a locally integrable function, $-2E_{0.5,1}(-2x_+^{0.5})$ is continuous over $[0, \infty)$ and $u \in \mathcal{D}'(\mathbb{R}^+)$.

1.5. Applications of Fractional Differential Equations in Distributions

Studying fractional differential equations (FDEs) in the distributional (Schwartz) sense is not just a formal generalization; it is essential in many settings where classical derivatives fail to exist or to capture singular behavior with the following main application domains.

(1) Modeling of Singular or Impulsive Sources: Many physical systems involve sources that are localized at points or interfaces, such as impulses, shocks, or discontinuities. In such settings, the right-hand side of a fractional differential equation (FDE) may be represented by a distribution; for example,

$$\frac{d^{1.5}}{dx^{1.5}} u(x) = \delta^{(0.5)}(x) \quad \text{or} \quad \frac{d^{0.5}}{dx^{0.5}} u(x) = \theta(x) + x\delta'(x).$$

To make sense of such equations, one must interpret the fractional derivatives $\frac{d^{1.5}}{dx^{1.5}}$ and $\frac{d^{0.5}}{dx^{0.5}}$ as acting on distributions. This gives consistent, mathematically rigorous definitions of the Green's functions for fractional operators.

(2) Viscoelasticity and Materials Science: Real-world materials like polymers, gels, and biological tissues exhibit behavior that is neither purely elastic (like a spring) nor purely viscous (like a dashpot) but somewhere in between. This is called viscoelasticity.

(a) Fractional Model: The stress–strain relationship is often modeled by fractional differential equations (e.g., using fractional Kelvin–Voigt or Zener models). The fractional order captures the "memory" of the material.

(b) Distributional Sense: What if the material is subjected to an impact load (a hammer strike)? This is modeled as a Dirac delta distribution, $\delta(x)$. To solve the FDE with this impulsive forcing term, one must work in the distributional framework. The solution will show how the material responds to a sudden, singular input.

(3) Signal Processing and System Identification: Many physical systems [8] are "fractional-order systems," meaning their transfer function involves fractional powers of the Laplace variable s. Examples include certain electrical circuits with fractance devices, electrochemical processes, and diffusion-wave phenomena. To analyze the response of such a system to an impulse (to find its impulse response or Green's function), the input is $\delta(x)$. The governing FDE is inherently distributional. This allows engineers to characterize systems with infinite speed of propagation or long-term memory that classical integer-order models cannot capture accurately.

(4) Regularization and Analytical Continuation: Fractional integrals Φ_α for Re $\alpha > 0$ act as regularizing operators on distributions. For example, if f is a distribution supported in $\mathbb{R}^+$, then $\Phi_\alpha * f$ becomes smoother. This is the basis of the Riemann–Liouville regularization technique, used to assign meaning to otherwise divergent or singular expressions.

We investigate the existence and uniqueness of the following generalized nonlinear Bagley–Torvik equation in $\mathcal{D}'(\mathbb{R}^+)$ for $0 < \alpha_1 < \cdots < \alpha_m < 2$ and constants b_i ($i = 0, 1, \cdots, m$):

$$\frac{d^2}{dx^2}y(x) + b_1 \frac{d^{\alpha_1}}{dx^{\alpha_1}}y(x) + \cdots + b_m \frac{d^{\alpha_m}}{dx^{\alpha_m}}y(x) + b_0\, y(x) = g(x, y(x)), \tag{10}$$

based on the inverse operator, the multivariate Mittag–Leffler function, Leray–Schauder's fixed-point theorem, and Banach's contractive principle. Finally, several examples are presented to demonstrate applications of our main theorems.

We should point out that neither initial nor boundary conditions are imposed on the equation here, since distributions do not possess well-defined pointwise values; for example, $\delta(0)$ has no meaning.

The generalized nonlinear Bagley–Torvik equation, involving multiple fractional derivatives of orders between zero and two, is a powerful model for describing systems with memory, hereditary effects, and complex damping behavior. Its broad mathematical framework allows it to capture phenomena that cannot be adequately represented by classical integer-order differential equations. Because fractional derivatives encode information about past states of a system, this equation is especially suited to modeling materials and processes with history-dependent responses.

In mechanical and structural engineering, the equation is used to describe viscoelastic materials and damped vibrations. The original Bagley–Torvik model arose in the study of a rigid plate immersed in a Newtonian fluid, where the fractional derivative represented a frequency-dependent damping force [9,10]. Its generalized nonlinear form now models beams, plates, and other structural components made of viscoelastic or composite materials, where traditional linear damping laws fail. Such models are widely applied in vibration control, aerospace engineering, and seismic design, where accurately capturing damping is essential for predicting long-term stability and resonance behavior.

In fluid mechanics, the fractional Bagley–Torvik equation appears in the modeling of non-Newtonian and viscoelastic fluids, where stress depends on the entire deformation history rather than the instantaneous rate of strain. This includes applications to polymeric liquids, electrorheological fluids, and biological fluids, all of which display anomalous stress relaxation and memory effects. Fractional derivatives allow the governing equations to bridge the gap between purely elastic and purely viscous behavior, providing a more realistic representation of such materials.

When the generalized nonlinear Bagley–Torvik equation is studied in the distributional setting, namely in the space of distributions $\mathcal{D}'(\mathbb{R}^+)$, its range of applicability expands considerably. Within this framework, both classical and fractional derivatives are understood in the sense of generalized derivatives, enabling the equation to accommodate singular data, impulsive sources, and non-smooth phenomena that naturally arise in many physical and engineering applications.

According to the authors' best knowledge, there is little to no research on the generalized nonlinear Bagley–Torvik equation in distributions. However, several studies have addressed this problem in the classical setting. For example, Liu et al. [11] proposed an improved numerical method for the fractional Bagley–Torvik equation with integral boundary conditions by transforming the original problem into a weakly singular Fredholm–Hammerstein integral equation of the second kind. Similarly, Aljazzazi et al. [12] investigated the effectiveness of the reproducing kernel Hilbert space method for obtaining approximate numerical solutions of a class of fractional Bagley–Torvik equations subject to integral boundary conditions.

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2. The Generalized Bagley–Torvik Equation

This section investigates the generalized linear Bagley–Torvik equation in the distributional space $\mathcal{D}'(\mathbb{R}^+)$, as well as the generalized nonlinear Bagley–Torvik Equation (10) in $C[0, T]$. The analysis employs the inverse-operator technique together with Banach's fixed-point theorem and is supplemented with several illustrative examples.

2.1. The Generalized Linear Bagley–Torvik Equation

Lemma 1. *Let $f \in \mathcal{D}'(\mathbb{R}^+)$. Then, the generalized linear Bagley–Torvik equation for $0 < \alpha_1 < \cdots < \alpha_m < 2$ and constants b_i $(i = 0, 1, \cdots, m)$,*

$$\frac{d^2}{dx^2}y(x) + b_1 \frac{d^{\alpha_1}}{dx^{\alpha_1}}y(x) + \cdots + b_m \frac{d^{\alpha_m}}{dx^{\alpha_m}}y(x) + b_0\, y(x) = f(x) \tag{11}$$

has a unique solution

$$y(x) = y_1(x) + y_2(x) \tag{12}$$

in $\mathcal{D}'(\mathbb{R}^+)$, where

$$y_1(x) = \sum_{k=0}^{k^*-1} (-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m} \cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f$$

is a distribution in $\mathcal{D}'(\mathbb{R}^+)$, and

$$y_2(x) = \sum_{k=k^*}^{\infty} (-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m} \cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f$$

is a continuous function in $[0, \infty)$, and the minimum $k^* \geq 0$ is chosen such that $k_0 + k_1 + \cdots + k_m = k^*$, and

$$\Phi_{\min\{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m\}+2} * f$$

is continuous over $[0, \infty)$. In particular, $k^* = 0$ if f is locally integrable.

Proof. Applying the distribution Φ_2 to both sides of Equation (11), we obtain

$$y(x) + b_1\, \Phi_{2-\alpha_1} * y(x) + \cdots + b_m\, \Phi_{2-\alpha_m} * y(x) + b_0 \Phi_2 * y(x) = \Phi_2 * f.$$

This implies that

$$(\delta + b_1\, \Phi_{2-\alpha_1} + \cdots + b_m\, \Phi_{2-\alpha_m} + b_0 \Phi_2) * y = \Phi_2 * f.$$

Define an operator $\mathcal{P}$ over $C[0, T]$ with an arbitrary $T > 0$ as

$$\begin{aligned} \mathcal{P} &= \sum_{k=0}^{\infty} (-1)^k (b_1\, \Phi_{2-\alpha_1} + \cdots + b_m\, \Phi_{2-\alpha_m} + b_0\, \Phi_2)^k \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m} \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m}, \end{aligned}$$

which is well-defined. In fact, for any $\psi \in C[0,T]$, we have

$$\begin{aligned}\|\mathcal{P} * \psi\| \leq & \|\psi\| \sum_{k=0}^{\infty} \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} |b_0|^{k_0} |b_1|^{k_1} \cdots |b_m|^{k_m} \\ & \cdot \frac{T^{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m}}{\Gamma(2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+1)} \\ & = \|\psi\| E_{(2,2-\alpha_1,\cdots,2-\alpha_m),1}\left(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\right) < +\infty.\end{aligned}$$

Since $T > 0$ is arbitrary, $\mathcal{P} * \psi$ is continuous over $[0,\infty)$ due to the above uniform convergence. Furthermore,

$$\begin{aligned}&\mathcal{P} * (\delta + b_1 \Phi_{2-\alpha_1} + \cdots + b_m \Phi_{2-\alpha_m} + b_0 \Phi_2) \\ &= (\delta + b_1 \Phi_{2-\alpha_1} + \cdots + b_m \Phi_{2-\alpha_m} + b_0 \Phi_2) * \mathcal{P} = \delta.\end{aligned}$$

Clearly,

$$\begin{aligned}&\mathcal{P} * (\delta + b_1 \Phi_{2-\alpha_1} + \cdots + b_m \Phi_{2-\alpha_m} + b_0 \Phi_2) \\ &= \mathcal{P} + \sum_{k=0}^{\infty} (-1)^k (b_1 \Phi_{2-\alpha_1} + \cdots + b_m \Phi_{2-\alpha_m} + b_0 \Phi_2)^{k+1} \\ &= \delta + \sum_{k=1}^{\infty} (-1)^k (b_1 \Phi_{2-\alpha_1} + \cdots + b_m \Phi_{2-\alpha_m} + b_0 \Phi_2)^{k} \\ &\quad + \sum_{k=0}^{\infty} (-1)^k (b_1 \Phi_{2-\alpha_1} + \cdots + b_m \Phi_{2-\alpha_m} + b_0 \Phi_2)^{k+1} \\ &= \delta.\end{aligned}$$

Similarly,

$$(\delta + b_1 \Phi_{2-\alpha_1} + \cdots + b_m \Phi_{2-\alpha_m} + b_0 \Phi_2) * \mathcal{P} = \delta,$$

and $\mathcal{P}$ is unique. In summary,

$$\begin{aligned}y(x) =& \mathcal{P} * (\Phi_2 * f) = \sum_{k=0}^{\infty} (-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m} \\ & \cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f.\end{aligned}$$

As $f \in \mathcal{D}'(\mathbb{R}^+)$, there exists a minimum nonnegative integer k^* such that

$$\Phi_{\min\{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m\}+2} * f$$

is continuous over $[0,\infty)$ (by noting that $\Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2}$ is a regularizing operator), where

$$k_0 + k_1 + \cdots + k_m = k^*.$$

For example, if $f = \delta'$ (or it is locally integrable), then we choose $k^* = 0$,

$$\Phi_2 * \delta' = \Phi_2 * \Phi_{-1} = \Phi_1 = \theta \quad \text{or} \quad (\Phi_2 * f),$$

since both are continuous over $[0,\infty)$. Therefore,

$$
\begin{aligned}
y(x) = &\sum_{k=0}^{k^*-1}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k}\binom{k}{k_0,k_1,\cdots,k_m} b_0^{k_0} b_1^{k_1}\cdots b_m^{k_m} \\
&\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f \\
&+ \sum_{k=k^*}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k}\binom{k}{k_0,k_1,\cdots,k_m} b_0^{k_0} b_1^{k_1}\cdots b_m^{k_m} \\
&\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f = y_1(x) + y_2(x),
\end{aligned}
$$

where

$$
\begin{aligned}
y_1(x) = &\sum_{k=0}^{k^*-1}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k}\binom{k}{k_0,k_1,\cdots,k_m} b_0^{k_0} b_1^{k_1}\cdots b_m^{k_m} \\
&\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f
\end{aligned}
$$

is a distribution in $\mathcal{D}'(\mathbb{R}^+)$ (the sum is interpreted as zero if $k^* = 0$), and

$$
\begin{aligned}
y_2(x) = &\sum_{k=k^*}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k}\binom{k}{k_0,k_1,\cdots,k_m} b_0^{k_0} b_1^{k_1}\cdots b_m^{k_m} \\
&\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f
\end{aligned}
$$

is a continuous on $[0,\infty)$. To see $y_2(x) \in C[0,\infty)$, we consider

$$
\begin{aligned}
y_2(x) = &(-1)^{k^*} \sum_{k_0+k_1+\cdots+k_m=k^*}\binom{k^*}{k_0,k_1,\cdots,k_m} b_0^{k_0} b_1^{k_1}\cdots b_m^{k_m} \\
&\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f \\
&+ \sum_{k=k^*+1}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k}\binom{k}{k_0,k_1,\cdots,k_m} b_0^{k_0} b_1^{k_1}\cdots b_m^{k_m} \\
&\cdot \Phi_{2k'_0+(2-\alpha_1)k'_1+\cdots+(2-\alpha_m)k'_m} * \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f \\
= &\, y_{21}(x) + y_{22}(x),
\end{aligned}
$$

where

$$
k'_0 + k'_1 + \cdots + k'_m \geq 1.
$$

Obviously, the finite sum

$$
\begin{aligned}
y_{21}(x) = &(-1)^{k^*} \sum_{k_0+k_1+\cdots+k_m=k^*}\binom{k^*}{k_0,k_1,\cdots,k_m} b_0^{k_0} b_1^{k_1}\cdots b_m^{k_m} \\
&\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f
\end{aligned}
$$

is a continuous function on $C[0,\infty)$. Let $k_0 + k_1 + \cdots + k_m = k^*$, and

$$
w = \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * f
$$

be a continuous function over $[0,\infty)$. Estimating the norm of y_{22} over an interval $[0,T]$, we get

$$
\begin{aligned}
\|y_{22}\| \leq &\max_{k_0+k_1+\cdots+k_m=k^*} \|w\| \sum_{k'=1}^{\infty} \sum_{k'_0+k_1+\cdots+k'_m=k'}\binom{k'}{k'_0,k'_1,\cdots,k'_m} |b_0|^{k'_0} b_1^{k'_1}\cdots |b_m|^{k'_m} \\
&\cdot \frac{T^{2k'_0+(2-\alpha_1)k'_1+\cdots+(2-\alpha_m)k'_m}}{\Gamma(2k'_0+(2-\alpha_1)k'_1+\cdots+(2-\alpha_m)k'_m+1)} < +\infty,
\end{aligned}
$$

by using the multivariate Mittag–Leffler function. Since $T > 0$ is arbitrary, y_{22} is continuous over $[0, \infty)$. This completes the proof. $\square$

Example 1. *The linear Bagley–Torvik equation*

$$\frac{d^2}{dx^2}y(x) - 3\frac{d^{1.5}}{dx^{1.5}}y(x) = \delta^{(2)}(x) \tag{13}$$

has a unique solution

$$y(x) = \delta(x) + \frac{3\,x_+^{-0.5}}{\sqrt{\pi}} + 9E_{0.5,1}(3x_+^{0.5})$$

in $\mathcal{D}'(\mathbb{R}^+)$.

Indeed, we derive that by applying the distribution Φ_2 to both sides of Equation (13):

$$y(x) - 3\Phi_{0.5} * y(x) = \Phi_2 * \delta^{(2)} = \delta(x).$$

This deduces that

$$\begin{aligned} y(x) = (\delta - 3\Phi_{0.5})^{-1} * \delta &= \sum_{k=0}^{\infty} 3^k \Phi_{0.5k} * \delta = \delta(x) + 3\,\frac{x_+^{-0.5}}{\Gamma(0.5)} + \sum_{k=2}^{\infty} 3^k \Phi_{0.5k} * \delta \\ &= \delta(x) + \frac{3\,x_+^{-0.5}}{\sqrt{\pi}} + \sum_{k=2}^{\infty} 3^k \frac{x_+^{0.5k-1}}{\Gamma(0.5k)} = \delta(x) + \frac{3\,x_+^{-0.5}}{\sqrt{\pi}} + 9\sum_{k=0}^{\infty} 3^k \frac{x_+^{0.5k}}{\Gamma(0.5k+1)} \\ &= \delta(x) + \frac{3\,x_+^{-0.5}}{\sqrt{\pi}} + 9E_{0.5,1}(3x_+^{0.5}), \end{aligned}$$

using

$$\begin{aligned} \sum_{k=2}^{\infty} 3^k \Phi_{0.5k} * \delta &= \sum_{k=2}^{\infty} 3^k \Phi_{0.5k} = \sum_{k=2}^{\infty} 3^k \frac{x_+^{0.5k-1}}{\Gamma(0.5k)} = \sum_{k=0}^{\infty} 3^{k+2} \frac{x_+^{0.5(k+2)-1}}{\Gamma(0.5(k+2))} \\ &= 9\sum_{k=0}^{\infty} 3^k \frac{x_+^{0.5k}}{\Gamma(0.5k+1)}. \end{aligned}$$

Clearly, $k^* = 1$ in Lemma 1, $y_1(x) = \delta(x) + \frac{3\,x_+^{-0.5}}{\sqrt{\pi}}$ is a distribution in $\mathcal{D}'(\mathbb{R}^+)$, and $y_2(x) = 9E_{0.5,1}(3x_+^{0.5})$ is a continuous function over $[0, \infty)$.

Remark 1. *(a) It seems impossible to solve Equation (11) using the Laplace transform when* $f \neq 0$ *is a distribution. For example, if* $f(x) = x_+^{-3/2}$*, then the Laplace transform of* f *does not exist. However, in the classical setting, the Laplace transform method is closely related to the inverse operator method for homogeneous equations with constant coefficients. More generally, the inverse operator method is considerably more powerful than the Laplace transform.*

(b) Following the above technique, we are able to study the generalized fractional differential equation in distributions for $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \beta$:

$$\frac{d^\beta}{dx^\beta}y(x) + b_1\,\frac{d^{\alpha_1}}{dx^{\alpha_1}}y(x) + \cdots + b_m\,\frac{d^{\alpha_m}}{dx^{\alpha_m}}y(x) + b_0\,y(x) = f(x),$$

where $f \in \mathcal{D}'(\mathbb{R}^+)$*. This can be done by applying the operator* Φ_β *to both sides of the equation and then using the inverse operator approach.*

This generalized multi-term fractional differential equation models processes with multiple interacting memory effects, each operating on a different time scale. Such systems arise naturally in complex viscoelastic materials, where several relaxation mechanisms coexist; in multirate anomalous *diffusion, where particles experience different trapping behaviors; and in mechanical vibrations with layered or composite damping, where each fractional term reflects a distinct hereditary component. The equation also appears in control theory and signal processing, where multi-term fractional operators describe systems with several feedback or filtering dynamics. Allowing the forcing f to be a distribution makes the model capable of handling impulses, shocks, and singular inputs, which occur in engineering, physics, and applied sciences.*

2.2. The Generalized Nonlinear Bagley–Torvik Equation

Theorem 1. *Let $T > 0$, g be a continuous function on $[0, T] \times \mathbb{R}$ and $g(x, 0)$ be bounded, satisfying the following Lipschitz condition for a nonnegative constant $\mathcal{L}$:*

$$|g(x, y_1) - g(x, y_2)| \le \mathcal{L}|y_1 - y_2|, \quad \textit{for all} \;\; y_1, y_2 \in \mathbb{R}.$$

In addition,

$$q = \mathcal{L}T^2 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\left(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\right) < 1.$$

Then, Equation (10) has a unique solution in $C[0, T]$.

Proof. We begin defining a nonlinear mapping $\mathcal{T}$ over $C[0, T]$ from Section 2.1 as

$$(\mathcal{T}y)(x) = \sum_{k=0}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m} \cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * g(x, y(x)),$$

which is a mapping from $C[0, T]$ to itself by noting that

$$|g(x, y(x))| \le |g(x, y(x)) - g(x, 0) + g(x, 0)| \le |g(x, y(x)) - g(x, 0)| + |g(x, 0)| \le \mathcal{L}|y(x)| + |g(x, 0)|,$$

which implies that for any fixed $y \in C[0, T]$,

$$\sup_{x\in[0,T]} |g(x, y(x))| \le \mathcal{L}\|y\| + \sup_{x\in[0,T]} |g(x, 0)| < +\infty,$$

since $g(x, 0)$ is bounded. It remains to be shown that $\mathcal{T}$ is contractive. Indeed, for $y_1, y_2 \in C[0, T]$,

$$(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x) = \sum_{k=0}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m} \cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * (g(x, y_1(x)) - g(x, y_2(x)),$$

by noting that

$$|\Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * (g(x, y_1(x)) - g(x, y_2(x))| \le \mathcal{L}\|y_1 - y_2\| \frac{T^{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2}}{\Gamma(2k_0 + (2-\alpha_1)k_1 + \cdots + (2-\alpha_m)k_m + 3)}.$$

Hence,

$$\|\mathcal{T}y_1 - \mathcal{T}y_2\| \le T^2 \mathcal{L} E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\left(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\right)\|y_1 - y_2\| = q\|y_1 - y_2\|,$$

where

$$q = \mathcal{L}T^2 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\left(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\right) < 1.$$

By Banach's contractive principle, Equation (10) has a unique solution in $C[0,T]$. This completes the proof. □

Example 2. *The nonlinear Bagley–Torvik equation*

$$y^{(2)}(x) + \frac{1}{9}\frac{d^{1.5}}{dx^{1.5}}y(x) + \frac{1}{5}y(x) = \frac{1}{40}\cos\left(\frac{y(x)}{x^2+1}\right) + \frac{1}{40}|y(x)| + \frac{1}{x^4+2}, \quad x \in \left[0, \sqrt{\pi}\,\right], \tag{14}$$

has a unique solution in $C\left[0, \sqrt{\pi}\,\right]$.

In fact,

$$g(x, y(x)) = \frac{1}{40}\cos\left(\frac{y(x)}{x^2+1}\right) + \frac{1}{40}|y(x)| + \frac{1}{x^4+2},$$

is a continuous function over $\left[0, \sqrt{\pi}\,\right]$ and $g(x,0)$ is bounded, satisfying the Lipschitz condition:

$$|g(x,\zeta_1) - g(x,\zeta_2)| \leq \frac{1}{40}\left|\frac{\zeta_1}{x^2+1} - \frac{\zeta_2}{x^2+1}\right| + \frac{1}{40}|\zeta_1 - \zeta_2| \leq \frac{1}{20}|\zeta_1 - \zeta_2|,$$

which implies that $\mathcal{L} = \dfrac{1}{20}$. Then, we need to evaluate the value of q for $m = 1$:

$$\begin{aligned} q &= \mathcal{L}T^2 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\left(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\right) \\ &= \frac{\pi}{20}E_{(2,0.5),3}\left(\frac{1}{5}\pi, \frac{1}{9}\pi^{1/4}\right) \approx \frac{\pi}{20} * 0.5792242101936123 < 1, \end{aligned}$$

by our Python (version 3) codes. Thus, Equation (14) has a unique solution in $C\left[0, \sqrt{\pi}\,\right]$.

Remark 2. *In general, assigning meaning to compositions of distributions is difficult and technically challenging [3]. For this reason, we investigate the uniqueness of Equation (10) within the framework of continuous functions under the distributional derivative sense.*

3. Existence

We use the following Leray–Schauder's fixed-point theorem to study the existence of solutions to Equation (10).

Theorem 2. *(Leray–Schauder's fixed-point theorem [13]) Let T be a continuous and compact mapping of a Banach space X to itself, such that the set $\{x \in X : x = \epsilon Tx$ for some $0 < \epsilon < 1\}$ is bounded. Then, T has a fixed point.*

Theorem 3. *Let $T > 0$ and g be a continuous function on $[0,T] \times \mathbb{R}$, satisfying*

$$|g(x,y)| \leq c_0 + c_1|y|, \quad (x,y) \in [0,T] \times \mathbb{R},$$

for some nonnegative constants c_0 and c_1. In addition, we assume that

$$Q = T^2 c_1 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\left(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\right) < 1.$$

Then, Equation (10) has at least one solution in $C[0,T]$.

Proof. We consider the nonlinear mapping $\mathcal{T}$ again over $C[0,T]$ by

$$(\mathcal{T}y)(x) = \sum_{k=0}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m}$$
$$\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * g(x, y(x)).$$

It follows that $\mathcal{T}$ is a mapping from $C[0,T]$ to itself. We are going to show that

(i) $\mathcal{T}$ is continuous. Indeed,

$$(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x) = \sum_{k=0}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m}$$
$$\cdot \Phi_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * (g(x, y_1(x)) - g(x, y_2(x))).$$

This implies that

$$\|\mathcal{T}y_1 - \mathcal{T}y_2\|$$
$$\leq T^2 \sup_{x\in[0,T]} |g(x, y_1(x)) - g(x, y_2(x))| E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\Big(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\Big),$$

from the proof of Theorem 1. It is clear that the continuity of g implies the continuity of the operator $\mathcal{T}$.

(ii) Further, we show that $\mathcal{T}$ is a mapping from a bounded set $W \subset C[0,T]$ to a bounded set in $C[0,T]$. This is clearly true since

$$\|\mathcal{T}y\| \leq T^2 \sup_{x\in[0,T]} |g(x, y(x))| E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\Big(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\Big)$$

is uniformly bounded if $y \in W$, as the term $\sup_{x\in[0,T]} |g(x, y(x))|$ is uniformly bounded.

(iii) We show that $\mathcal{T}$ is completely continuous from $C[0,T]$ to itself. Then, using the Arzela–Ascoli theorem, we only need to prove that $\mathcal{T}$ is equicontinuous on every bounded set W of $C[0,T]$. To proceed with this, we let $0 \leq x_1 < x_2 \leq T$ and $y \in W$, and consider

$$(\mathcal{T}y)(x_2) - (\mathcal{T}y)(x_1) = \sum_{k=0}^{\infty}(-1)^k \sum_{k_0+k_1+\cdots+k_m=k} \binom{k}{k_0, k_1, \cdots, k_m} b_0^{k_0} b_1^{k_1} \cdots b_m^{k_m}$$
$$\cdot \Big(\Phi^{x=x_2}_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * g(x, y(x)) - \Phi^{x=x_1}_{2k_0+(2-\alpha_1)k_1+\cdots+(2-\alpha_m)k_m+2} * g(x, y(x))\Big).$$

Let

$$K = 2k_0 + (2-\alpha_1)k_1 + \cdots + (2-\alpha_m)k_m + 2.$$

Clearly,

$$\Phi_K^{x=x_2} * g(x, y(x)) - \Phi_K^{x=x_1} * g(x, y(x))$$
$$= \frac{1}{\Gamma(K)} \int_0^{x_1} ((x_2 - t)^{K-1} - (x_1 - t)^{K-1}) g(t, y(t)) dt + \frac{1}{\Gamma(K)} \int_{x_1}^{x_2} (x_2 - t)^{K-1} g(t, y(t)) dt$$
$$= I_1 + I_2.$$

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Regarding I_1,

$$\begin{aligned}|I_1| &\le \sup_{t\in[0,T]} |g(t,y(t))| \frac{1}{\Gamma(K)} \int_0^{x_1} ((x_2-t)^{K-1} - (x_1-t)^{K-1})dt \\ &= \sup_{t\in[0,T]} |g(t,y(t))| \frac{1}{\Gamma(K+1)} (-(x_2-x_1)^K + x_2^K - x_1^K) \\ &\le \sup_{t\in[0,T]} |g(t,y(t))| \frac{1}{\Gamma(K+1)} (x_2^K - x_1^K), \quad \text{since } x_1 < x_2,\end{aligned}$$

which contains the factor $x_2 - x_1$ by the mean value theorem. In fact,

$$\frac{x_2^K - x_1^K}{x_2 - x_1} = \theta^{K-1}, \quad \theta \in (x_1, x_2)$$

which claims that

$$|x_2^K - x_1^K| \le T^{K-1}(x_2 - x_1).$$

Thus, it is equicontinuous.

As for I_2,

$$|I_2| \le \sup_{t\in[0,T]} |g(t,y(t))| \frac{1}{\Gamma(K)} \int_{x_1}^{x_2} (x_2-t)^{K-1} dt \le \sup_{t\in[0,T]} |g(t,y(t))| \frac{1}{\Gamma(K)} T^{K-1}(x_2 - x_1),$$

by noting that $(x_2-t)^{K-1} \le T^{K-1}$, which is also equicontinuous. Hence, $\mathcal{T}$ is a compact operator by the Arzela–Ascoli theorem.

(iv) The set

$$\{y \in C[0,T] : y = \epsilon \mathcal{T} y \text{ for some } 0 < \epsilon < 1\}$$

is uniformly bounded. This is obviously true since

$$\begin{aligned}&\|y\| < \|\mathcal{T}y\| \\ &\le T^2 \sup_{x\in[0,T]} |g(x,y(x))| E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\Big(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\Big) \\ &\le T^2 c_0 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\Big(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\Big) \\ &+ T^2 c_1 \|y\| E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\Big(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\Big).\end{aligned}$$

Since

$$Q = T^2 c_1 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\Big(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\Big) < 1,$$

this deduces that

$$\|y\| < \frac{1}{1-Q} T^2 c_0 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\Big(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\Big)$$

is bounded. By Leray–Schauder's fixed-point theorem, Equation (10) has at least one solution in $C[0,T]$. This completes the proof. □

Example 3. *The nonlinear Bagley–Torvik equation*

$$y^{(2)}(x) + \frac{5}{7} \frac{d^{0.6}}{dx^{0.6}} y(x) + \frac{2}{5} y(x) = \frac{1}{8} \cos\left(\frac{y^3(x)}{y^2(x)+1}\right) + \frac{1}{12}|y(x)| + \frac{x^4+2}{|y(x)|+1}, \quad x \in [0,3], \tag{15}$$

has at least one solution in $C[0,3]$.

It follows that

$$g(x,y) = \frac{1}{8}\cos\left(\frac{y^3}{y^2+1}\right) + \frac{1}{12}|y| + \frac{x^4+2}{|y|+1}$$

is a continuous function on $[0,3] \times \mathbb{R}$, satisfying

$$|g(x,y)| \leq 84 + \frac{1}{12}|y|.$$

Furthermore,

$$\begin{aligned} Q &= T^2 c_1 E_{(2,2-\alpha_1,\cdots,2-\alpha_m),3}\left(|b_0|T^2, |b_1|T^{2-\alpha_1}, \cdots, |b_m|T^{2-\alpha_m}\right) \\ &= \frac{9}{12}E_{(2,1.4),3}\left(18/5, 5/7 * 3^{1.4}\right) \approx \frac{11.146190691515729806}{12} < 1. \end{aligned}$$

By Theorem 3, the equation has at least one solution in $C[0,3]$.

Remark 3. *It would be worthwhile to consider the following fractional differential equation with a variable coefficient:*

$$\frac{d^\beta}{dx^\beta}y(x) + b(x) * \frac{d^\alpha}{dx^\alpha}y(x) = f(x),$$

where $b \in C[0,\infty)$ *and* $f \in \mathcal{D}'(\mathbb{R}^+)$.

This type of fractional differential equation is useful for modeling systems that exhibit memory effects while also having properties that change over time or space. The variable coefficient $b(x)$ *allows the strength of the memory term to vary, making the equation suitable for describing heterogeneous or time-dependent materials in viscoelasticity, nonuniform anomalous diffusion processes, and mechanical systems with changing damping. Because the forcing term* f *may be a distribution, the equation can also handle impulsive or singular inputs, which occur in control theory, signal processing, and models involving sudden shocks or loads.*

4. Conclusions

We developed a theory of fractional calculus for distributions supported on $\mathbb{R}^+$ in the sense of Schwartz, with distributional convolution serving as the fundamental analytical tool. Building on this framework, we investigated both the linear and nonlinear generalized Bagley–Torvik equations by employing inverse operators, the multivariate Mittag–Leffler function, Banach's contraction principle, and the Leray–Schauder fixed-point theorem. Several illustrative examples were also provided to demonstrate the applicability of the theory. Moreover, the proposed approach can be extended to a broad class of differential equations.

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