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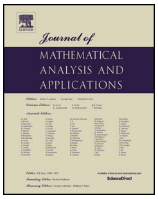

Regular Articles

Blow-up and core transition for mass-constrained nonlinear Schrödinger equations with combined nonlinearities

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ABSTRACT

We investigate mass-constrained ground states for nonlinear Schrödinger equations with combined focusing power nonlinearities in the mass-subcritical regime $2 < q < p < 2 + \frac{4}{N}$. The problem is formulated in $\mathbb{R}^N$ under a confining potential possessing finitely many isolated zero wells, and ground states are characterized as minimizers of the energy functional subject to a fixed L^2 constraint.

We examine the qualitative behavior of minimizers as the prescribed mass varies. In the small-mass regime, ground states bifurcate from the first eigenfunction of the associated linear Schrödinger operator and concentrate at its unique maximum point. In the large-mass regime, under a sharp condition on the interaction parameter, solutions exhibit blow-up behavior and converge, after suitable rescaling, to the autonomous ground state corresponding to the higher-power nonlinearity.

Furthermore, we identify a transition between a q-core and a p-core regime, determined by an intrinsic mass scale depending on the interaction parameter. The competition between the nonlinearities interacts with the geometry of the potential wells, giving rise to a mass-induced spatial phase transition in the localization of ground states.

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1. Problem statement

In this paper, we are interested in studying the following PDE with an integral condition:

$$\begin{cases} -\Delta u + V(x)u = \lambda u + \beta u^{p-1} + u^{q-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} |u|^2 \, dx = a > 0, \end{cases} \tag{1}$$

where $N \geq 1$, $2 < q < p < 2 + \frac{4}{N}$, $\beta > 0$, $\lambda \in \mathbb{R}$ is an unknown Lagrange multiplier, and the potential $V : \mathbb{R}^N \to [0, +\infty)$ will be defined later. We will work on the space

$$H^1_V(\mathbb{R}^N) := \left\{ u \in H^1(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} V(x)|u|^2 \, dx < +\infty \right\},$$

where $V : \mathbb{R}^N \to [0, +\infty)$ is a positive and continuous potential function. We consider the norm

$$\|u\|_{H^1_V(\mathbb{R}^N)} := \left(\int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2 + V(x)|u|^2 \right) dx \right)^{\frac{1}{2}}.$$

In addition, we define the energy functional

$$E(u) := \frac{1}{2} \int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)|u|^2 \right) dx - \frac{\beta}{p} \int\limits_{\mathbb{R}^N} |u|^p \, dx - \frac{1}{q} \int\limits_{\mathbb{R}^N} |u|^q \, dx, \tag{2}$$

with the mass constraint

$$\int\limits_{\mathbb{R}^N} |u|^2 \, dx = a > 0, \tag{3}$$

and the minimization problem

$$e(a) := \inf_{u \in S(a)} E(u) \tag{4}$$

where

$$S(a) := \left\{ u \in H^1_V(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} |u|^2 \, dx = a \right\}.$$

We assume that the potential V satisfies the following conditions:

- $(V1)$ $V \in C^2(\mathbb{R}^N)$, $V \geq 0$, and $\lim_{|x| \to +\infty} V(x) = +\infty$.
- $(V2)$ The zero set is finite and consists of the isolated points:

$$\mathcal{Z} := \{x \in \mathbb{R}^N : V(x) = 0\} = \{x_1, \ldots, x_m\}.$$

$(V3)$ For each $j \in \{1, \dots, m\}$ there exist $c_j > 0$ and $l_j \geq 2$ such that

$$V(x) = c_j|x - x_j|^{l_j} + o(|x - x_j|^{l_j}) \quad \text{as } x \to x_j. \tag{5}$$

For $2 < q < p < 2 + \frac{4}{N}$, let Q_p be the positive autonomous ground state of the following problem

$$-\Delta Q_p + Q_p = Q_p^{p-1}, \tag{6}$$

and $\|Q_p\|_2^2 = m_p$. Note that Q_p satisfies the Nehari and Pohozaev identities, which yield

$$\|\nabla Q_p\|_2^2 = \gamma_p \, \|Q_p\|_p^p, \qquad \|Q_p\|_2^2 = (1 - \gamma_p)\|Q_p\|_p^p,$$

where $\gamma_p := N\left(\frac{1}{2} - \frac{1}{p}\right)$. In particular,

$$\|\nabla Q_p\|_2^2 + \|Q_p\|_2^2 = \|Q_p\|_p^p.$$

For a minimizer u_a of E, we denote by $\mu_a := -\lambda_a$ and $\varepsilon_a := \mu_a^{-1/2}$. The right blow-up rescaling that we consider is

$$w_a(x) := \mu_a^{-\frac{1}{p-2}} u_a(x_a + \varepsilon_a x),$$

where $x_a \in \mathbb{R}^N$ is a maximum point. Then the rescaled equation becomes

$$-\Delta w_a + \mu_a^{-1} V(x_a + \varepsilon_a x) w_a = -w_a + \beta w_a^{p-1} + \mu_a^{-\delta} w_a^{q-1}, \tag{7}$$

where $\delta = \frac{p-q}{p-2}$. Also, let Q_q be the positive autonomous ground state of the following problem

$$-\Delta Q_q + Q_q = Q_q^{q-1}, \tag{8}$$

and $\|Q_q\|_2^2 = m_q$. For a minimizer u_a of E, we denote by $\mu_a := -\lambda_a$ and $\varepsilon_a := \mu_a^{-1/2}$. The right blow-up rescaling that we consider is

$$z_a(x) := \mu_a^{-\frac{1}{q-2}} u_a(x_a + \varepsilon_a x),$$

where $x_a \in \mathbb{R}^N$ is a maximum point. Then the rescaled equation becomes

$$-\Delta z_a + \mu_a^{-1} V(x_a + \varepsilon_a x) z_a = -z_a + \beta \mu_a^{-\gamma} z_a^{p-1} + z_a^{q-1}, \tag{9}$$

where $\gamma = \frac{p-q}{q-2}$. Let $H := -\Delta + V$. From $(V1) - (V3)$, we can assume that H has a ground eigenpair (ω_1, ϕ_1), $\phi_1 > 0$, $\|\phi_1\|_2 = 1$, $H\phi_1 = \omega_1 \phi_1$ and ϕ_1 has a unique global maximum point $x_1 \in \mathcal{Z}$. We define the zero-potential (autonomous) energy functional

$$E_0(u) := \frac{1}{2} \int\limits_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\beta}{p} \int\limits_{\mathbb{R}^N} |u|^p \, dx - \frac{1}{q} \int\limits_{\mathbb{R}^N} |u|^q \, dx, \qquad u \in H^1(\mathbb{R}^N).$$

For $a > 0$, we set

$$e_0(a) := \inf\Big\{E_0(u) : u \in H^1(\mathbb{R}^N),\ \|u\|_2^2 = a\Big\}.$$

Lemma 1.1. *[18] Let $u \in H^1(\mathbb{R}^N)$ and let $2 < r < \frac{2N}{N-2}$. Then the following Gagliardo–Nirenberg inequality holds:*

$$\|u\|_r \leq \left(\frac{r}{2\|U_r\|_2^{r-2}} \right)^{\frac{1}{r}} \|\nabla u\|_2^{\gamma_r} \, \|u\|_2^{1-\gamma_r}, \tag{10}$$

where $\gamma_r = \dfrac{N(r-2)}{2r}$. Moreover, up to translations, the function $U_r \in H^1(\mathbb{R}^N)$ is a ground state solution of

$$-\frac{N(r-2)}{4} \Delta U_r + \left(1 - \frac{(N-2)(r-2)}{4} \right) U_r = |U_r|^{r-2} U_r \quad \text{in } \mathbb{R}^N. \tag{11}$$

Lemma 1.2. *Assume that the potential V satisfies $(V1)$. Then the embedding*

$$H^1_V(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N),$$

is compact for every $2 \leq r < 2^*$.

Proof. Let $(u_n)_{n \geq 1} \subset H^1_V(\mathbb{R}^N)$ be a bounded sequence. Then (u_n) is bounded in $H^1(\mathbb{R}^N)$, since

$$\int\limits_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |u_n|^2 \right) dx \leq \|u_n\|^2_{H^1_V} \leq C.$$

Hence, up to a subsequence (not relabeled), there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^N), \qquad u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Moreover, by the Sobolev embedding, (u_n) is bounded in $L^{2^*}(\mathbb{R}^N)$ (if $N \geq 3$), and in $L^q(\mathbb{R}^N)$ for every $q \in [2, \infty)$ (if $N = 1, 2$). In particular, for every fixed $r \in [2, 2^*)$, there exists $C_r > 0$ such that

$$\sup_n \|u_n\|_{L^{2^*}(\mathbb{R}^N)} \leq C_r \quad (N \geq 3), \tag{12}$$

and similarly $\sup_n \|u_n\|_{L^q} \leq C_q$ for each $q < \infty$ when $N \leq 2$. Since (u_n) is bounded in H^1_V, we have

$$\sup_n \int\limits_{\mathbb{R}^N} V(x) \, |u_n|^2 \, dx \leq C. \tag{13}$$

Fix $\varepsilon > 0$. By the assumption $V(x) \to +\infty$ as $|x| \to \infty$, there exists $R > 0$ such that

$$V(x) \geq \frac{C}{\varepsilon} \qquad \text{for all } |x| \geq R.$$

Therefore, using (13),

$$\int\limits_{|x| \geq R} |u_n|^2 \, dx \leq \frac{\varepsilon}{C} \int\limits_{|x| \geq R} V(x) \, |u_n|^2 \, dx \leq \varepsilon \qquad \text{for all } n.$$

Hence,

$$\sup_n \int_{\mathbb{R}^N\setminus B_R} |u_n|^2\,dx \le \varepsilon. \tag{14}$$

By Rellich–Kondrachov on the ball B_R, for every $2 \le r < 2^*$ we have $u_n \to u$ strongly in $L^r(B_R)$. In particular,

$$\|u_n - u\|_{L^r(B_R)} \to 0 \quad \text{as } n \to \infty. \tag{15}$$

We claim that for every $2 < r < 2^*$,

$$\limsup_{n\to\infty} \|u_n - u\|_{L^r(\mathbb{R}^N\setminus B_R)} \le C\,\varepsilon^{\theta/2}, \tag{16}$$

for some $\theta \in (0,1)$ and a constant C independent of n and ε. Assume first that $N \ge 3$ and $2 < r < 2^*$. Choose $\theta \in (0,1)$ such that

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{2^*}.$$

By the interpolation (Hölder) inequality,

$$\|w\|_{L^r(\mathbb{R}^N\setminus B_R)} \le \|w\|^{\theta}_{L^2(\mathbb{R}^N\setminus B_R)} \|w\|^{1-\theta}_{L^{2^*}(\mathbb{R}^N\setminus B_R)} \qquad \text{for all } w.$$

Apply this to $w = u_n - u$. Since (u_n) is bounded in $H^1(\mathbb{R}^N)$ and $u \in H^1(\mathbb{R}^N)$, we have $\sup_n \|u_n - u\|_{L^{2^*}(\mathbb{R}^N)} \le C$ by (12). Moreover, by (14) and Fatou's lemma,

$$\int_{\mathbb{R}^N\setminus B_R} |u|^2\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} |u_n|^2\,dx \le \varepsilon.$$

Hence

$$\|u_n - u\|_{L^2(\mathbb{R}^N\setminus B_R)} \le \|u_n\|_{L^2(\mathbb{R}^N\setminus B_R)} + \|u\|_{L^2(\mathbb{R}^N\setminus B_R)} \le \varepsilon^{1/2} + \varepsilon^{1/2} = 2\varepsilon^{1/2}.$$

Therefore,

$$\|u_n - u\|_{L^r(\mathbb{R}^N\setminus B_R)} \le (2\varepsilon^{1/2})^{\theta}\, C^{1-\theta} = C'\,\varepsilon^{\theta/2},$$

which yields (16). When $N = 1, 2$, fix any $q > r$ and repeat the same argument using the boundedness in $L^q(\mathbb{R}^N)$ instead of L^{2^*} (Sobolev gives $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $q < \infty$). Choose $\theta \in (0,1)$ such that $\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{q}$.
Finally, for $r = 2$ we use (14) together with the local strong convergence in $L^2(B_R)$. Combining (15) and (16), we obtain

$$\|u_n - u\|_{L^r(\mathbb{R}^N)} \le \|u_n - u\|_{L^r(B_R)} + \|u_n - u\|_{L^r(\mathbb{R}^N\setminus B_R)} \to 0 \quad \text{as } n \to \infty,$$

since $\varepsilon > 0$ is arbitrary. Thus $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$, and the embedding is compact. □

2. Main results

Theorem 2.1 *(Small-mass regime). Assume that V satisfies $(V1)$–$(V3)$ and that $H = -\Delta + V$ has a simple first eigenvalue ω_1 with positive L^2–normalized eigenfunction ϕ_1, whose global maximum x_1 is unique. Let $2 < q < p < 2 + \frac{4}{N}$ and $\beta > 0$. For each $a > 0$, let u_a be a minimizer of $e(a)$ with $\|u_a\|_2^2 = a$, and let λ_a be the associated multiplier.*

Then, as $a \to 0$,

$$u_a = a\,\phi_1 + o(a) \quad \textit{in } H^1_V(\mathbb{R}^N),$$

and

$$\lambda_a = \omega_1 - \beta c_p a^{p-2} - c_q a^{q-2} + o(a^{p-2}),$$

where $c_p = \int \phi_1^p$ and $c_q = \int \phi_1^q$. Moreover, any maximum point x_a of u_a satisfies

$$x_a \to x_1.$$

Theorem 2.2 *(Large-mass regime). Assume $N \geq 1$, $2 < q < p < 2 + \frac{4}{N}$, and $\beta > 0$. Let V satisfy $(V1)$–$(V3)$ and assume*

$$\beta > \frac{p\gamma_p \|Q_p\|_2^{p-2}}{2},$$

where Q_p solves $-\Delta Q + Q = Q^{p-1}$. Let u_a be a minimizer of $e(a)$ and define

$$\mu_a := -\lambda_a, \qquad \varepsilon_a := \mu_a^{-1/2}.$$

Let x_a be a maximum point of u_a. Then $\lambda_a \to -\infty$ as $a \to +\infty$ and, defining

$$w_a(x) := \mu_a^{-\frac{1}{p-2}} u_a(x_a + \varepsilon_a x),$$

we have, up to a subsequence,

$$w_a \to \beta^{-\frac{1}{p-2}} Q_p \quad \textit{strongly in } H^1(\mathbb{R}^N).$$

Moreover,

$$\mu_a = \Big(\frac{\beta^{\frac{2}{p-2}} m_p}{a}\Big)^{\frac{p-2}{2-p\gamma_p}} (1 + o(1)),$$

and

$$\mathrm{dist}(x_a, \mathcal{Z}) \to 0.$$

If $\ell_* = \max_j \ell_j$ and $\mathcal{J}_* = \{j : \ell_j = \ell_*\}$, then every accumulation point of $\{x_a\}$ belongs to $\{x_j : j \in \mathcal{J}_*\}$. If the minimizer of $c_j \mathcal{M}_{\ell_j}$ over $\mathcal{J}_*$ is unique, then $x_a \to x_{j_*}$.

Theorem 2.3 *(Core transition). Assume $N \geq 1$, $2 < q < p < 2 + \frac{4}{N}$, $\beta > 0$, and $(V1)$–$(V3)$. There exists a mass scale $a_\times(\beta) > 0$ such that:*

(i) *(q-core) If $a \ll a_\times(\beta)$, then*

$$z_a(x) := \mu_a^{-\frac{1}{q-2}} u_a(x_a + \varepsilon_a x) \to Q_q \quad \text{in } H^1(\mathbb{R}^N).$$

(ii) *(p-core) If $a \gg a_\times(\beta)$ and $\beta > p\gamma_p \|Q_p\|_2^{p-2}$, then*

$$w_a(x) := \mu_a^{-\frac{1}{p-2}} u_a(x_a + \varepsilon_a x) \to \beta^{-\frac{1}{p-2}} Q_p \quad \text{in } H^1(\mathbb{R}^N).$$

(iii) *The crossover satisfies*

$$a_\times(\beta) \asymp \beta^{\frac{4-N(q-2)}{2(p-q)}}.$$

If $x_a \to x_j$, then

$$\int_{\mathbb{R}^N} V|u_a|^2 = c_j\, \varepsilon_a^{\ell_j} \int_{\mathbb{R}^N} |x|^{\ell_j} |U_a|^2 + o(\varepsilon_a^{\ell_j}),$$

with $U_a = z_a$ in the q-core and $U_a = w_a$ in the p-core. This describes how the nonlinear core transition affects well selection.

3. Background and motivation

The nonlinear Schrödinger equation

$$i\partial_t \psi = -\Delta\psi + V(x)\psi - g(|\psi|^2)\psi, \tag{17}$$

and its stationary counterpart

$$-\Delta u + V(x)u = \lambda u + g(u^2)u, \tag{18}$$

arise in nonlinear optics, Bose–Einstein condensation, plasma physics, and related fields. A natural class of solutions consists of functions with prescribed mass

$$\int_{\mathbb{R}^N} |u|^2\, dx = a > 0, \tag{19}$$

where $\lambda \in \mathbb{R}$ acts as a Lagrange multiplier. Such solutions, called *normalized ground states*, are obtained by minimizing the associated energy under the L^2 constraint. Understanding how these ground states behave as the mass a varies is a fundamental problem.

3.1. Mathematical background

For the pure-power nonlinearity $g(u) = |u|^{p-2}u$ in the mass-subcritical range $2 < p < 2 + \frac{4}{N}$, the theory is classical. Weinstein [18] established the sharp Gagliardo–Nirenberg inequality and characterized ground states as rescalings of the unique positive radial solution Q_p of $-\Delta Q + Q = Q^{p-1}$, whose uniqueness was proved by Kwong [14]. In the autonomous case $V \equiv 0$, existence, uniqueness, and qualitative properties are well understood; see Cazenave [6] and Lions [15,16].

When a confining potential V is present, the interaction between the potential and the nonlinearity leads to concentration phenomena. Byeon and Wang [4,5] studied the single-power equation under the assumption that V has finitely many isolated zeros with polynomial degeneracy $V(x) \sim c_j|x - x_j|^{\ell_j}$. They showed that minimizers concentrate at the maximum point of the first eigenfunction for small mass, while for large mass they localize at the flattest well and, among equally flat wells, at the one with the minimal coefficient. Related concentration results were developed by Ambrosetti–Malchiodi–Secchi [1], del Pino–Felmer [7], and Kang–Wei [13].

Normalized solutions under an L^2 constraint were systematically studied by Jeanjean [10], who introduced a variational framework restoring compactness. Further developments include Jeanjean–Le Coz [11], Bartsch–de Valeriola [2], Bartsch–Jeanjean–Soave [3], and Jeanjean–Lu [12].

3.2. Combined nonlinearities

Consider now the case of two competing focusing powers,

$$g(u) = \beta|u|^{p-2}u + |u|^{q-2}u, \qquad 2 < q < p < 2 + \frac{4}{N},\ \beta > 0. \tag{20}$$

Such nonlinearities arise in models with mixed interactions or higher-order corrections. In the autonomous case $V \equiv 0$, Soave [17] provided a classification of normalized ground states, highlighting the competition between the two powers. Additional results on existence, symmetry, and stability were obtained by Gou–Jeanjean [9]. The non-autonomous problem with a confining potential, however, remains largely open.

3.3. Applications of equation (1)

Equation (1) arises in several physical models described by nonlinear Schrödinger-type equations with a prescribed L^2 norm. In the theory of Bose–Einstein condensates, the function u represents the macroscopic wavefunction of a condensate confined by an external trapping potential $V(x)$, while the mass constraint $\int_{\mathbb{R}^N} |u|^2 dx = a$ corresponds to a fixed number of particles and the Lagrange multiplier λ is the associated chemical potential. The presence of combined focusing power nonlinearities reflects the coexistence of interactions acting at different density scales, such as effective higher-order or multi-body interactions beyond the standard cubic mean-field term. In this setting, different nonlinear powers may dominate in different mass regimes, leading to transitions between low-density and high-density localized states.

Similar equations also appear in nonlinear optics, where u describes the envelope of a laser beam propagating in a nonlinear medium with an inhomogeneous refractive index profile modeled by $V(x)$, and the L^2 constraint represents the conserved beam power. Media with mixed nonlinear response, such as Kerr-type and higher-order or saturating nonlinearities, naturally lead to combined-power models of the form (1). The resulting stationary solutions correspond to self-trapped optical beams (spatial solitons), whose localization and stability depend on the total power.

More generally, nonlinear Schrödinger equations with confining potentials possessing multiple wells provide canonical models for localization phenomena in structured media, including multi-trap condensates, optical lattices, and arrays of waveguides. In such systems, the interplay between the nonlinear interactions and the geometry of the potential may produce symmetry breaking and relocation of ground states between wells as the mass varies. Consequently, problem (1) serves as a prototypical framework for studying how competing nonlinearities and external confinement jointly determine the existence, concentration, and spatial localization of mass-constrained ground states.

3.4. Main contributions

In this paper, we provide a complete answer to the above questions. Our main contributions can be summarized as follows.

(i) **Small-mass regime.** We prove that as $a \to 0$, minimizers bifurcate from the first linear eigenfunction:

$$u_a = a\phi_1 + o(a) \quad \text{in } H^1_V(\mathbb{R}^N), \qquad \lambda_a = \omega_1 - \beta c_p a^{p-2} - c_q a^{q-2} + o(a^{p-2}),$$

where $c_p = \int \phi_1^p$, $c_q = \int \phi_1^q$, and $x_a \to x_1$ (the unique maximum point of ϕ_1). This extends the classical linearization argument to combined nonlinearities.

(ii) **Large-mass regime and blow-up analysis.** Under the sharp condition $\beta > p\gamma_p\|Q_p\|_2^{p-2}$ (with $\gamma_p = N(\frac{1}{2} - \frac{1}{p})$), we prove that $\mu_a := -\lambda_a \to \infty$ and the rescaling

$$w_a(x) := \mu_a^{-\frac{1}{p-2}} u_a(x_a + \epsilon_a x), \qquad \epsilon_a = \mu_a^{-1/2},$$

converges strongly in $H^1(\mathbb{R}^N)$ to $\beta^{-\frac{1}{p-2}} Q_p$. Moreover, we obtain the precise asymptotics

$$\mu_a = \left(\frac{\beta^{\frac{2}{p-2}} \|Q_p\|_2^2}{a} \right)^{\frac{p-2}{2-p\gamma_p}} (1 + o(1)) \qquad (a \to \infty).$$

(iii) **q-core and p-core regimes with explicit crossover scale.** We discover two distinct blow-up regimes:

- a q-core regime for $a \ll a_\times(\beta)$, where the q-normalized profile $z_a(x) := \mu_a^{-\frac{1}{q-2}} u_a(x_a + \epsilon_a x)$ converges to Q_q;
- a p-core regime for $a \gg a_\times(\beta)$, where the p-normalized profile $w_a(x) := \mu_a^{-\frac{1}{p-2}} u_a(x_a + \epsilon_a x)$ converges to $\beta^{-\frac{1}{p-2}} Q_p$;
- an explicit crossover scale $a_\times(\beta) \asymp \beta^{\frac{4-N(q-2)}{2(p-q)}}$ separating the two regimes.

This explicitly identifies how the competition between the two nonlinearities is resolved.

Notation

Throughout the paper, we denote by $\|\cdot\|_r$ the usual $L^r(\mathbb{R}^N)$ norm. The space $H^1_V(\mathbb{R}^N)$ is defined in Section 1. We write $A \lesssim B$ if $A \le C\,B$ for some constant $C > 0$ independent of the relevant parameters, and $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$ hold. The symbols $o(1)$ and $O(1)$ denote quantities that tend to zero or remain bounded, respectively, in the limit under consideration.

4. Proofs

Existence and basic structure

Lemma 4.1. *For every $a > 0$ we have $e(a) \in \mathbb{R}$, and $e(a)$ is achieved by a minimizer $u_a \in H^1_V(\mathbb{R}^N)$.*

Proof. By (2) and (10), we can write

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)|u|^2 \right) dx - \frac{\beta}{p} \int_{\mathbb{R}^N} |u|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q \, dx$$

$$\begin{aligned}
&\geq \frac{1}{2}\|u\|^2_{H^1_V(\mathbb{R}^N)} - \frac{\beta a^{p-p\gamma_p}}{2\|U_p\|_2^{p-2}}\|\nabla u\|_2^{p\gamma_p} - \frac{a^{q-q\gamma_q}}{2\|U_q\|_2^{q-2}}\|\nabla u\|_2^{q\gamma_q} \\
&\geq \frac{1}{2}\|u\|^2_{H^1_V(\mathbb{R}^N)} - \frac{\beta a^{p-p\gamma_p}}{2\|U_p\|_2^{p-2}}\|u\|^{p\gamma_p}_{H^1_V(\mathbb{R}^N)} - \frac{a^{q-q\gamma_q}}{2\|U_q\|_2^{q-2}}\|u\|^{q\gamma_q}_{H^1_V(\mathbb{R}^N)}. \qquad (21)
\end{aligned}$$

From (21), since $2 < q < p < 2 + \frac{4}{N}$, then $0 < q\gamma_q < p\gamma_p < 2$. Therefore, for any $a > 0$ and $u \in H^1_V(\mathbb{R}^N)$, $E(u) > -\infty$. Consequently $e(a) \in \mathbb{R}$, for all $a > 0$. Since energy is finite, Ekeland's variational principle [8] gives a sequence $(u_n)_{n\geq 1} \in H^1_V(\mathbb{R}^N)$ such that $E(u_n) \to e(a)$ and $E'(u_n) \to 0$ as $n \to +\infty$, where

$$\begin{aligned}
E'(u)[v] = &\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, dx + \int_{\mathbb{R}^N} V(x) u \cdot v\, dx \\
&- \lambda \int_{\mathbb{R}^N} u \cdot v\, dx - \beta \int_{\mathbb{R}^N} |u|^{p-2}uv\, dx - \int_{\mathbb{R}^N} |u|^{q-2}uv\, dx, \qquad (22)
\end{aligned}$$

where $v \in H^1_V(\mathbb{R}^N)$ is a test function. Hence, up to a subsequence (not relabeled), there exists $u_a \in H^1_V(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1_V(\mathbb{R}^N), \quad u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N, \quad u_n \to u \quad \text{in } L^{\mathrm{loc}}_r(\mathbb{R}^N),\, 2 \leq r < 2^*.$$

By (21), we conclude that the sequence $(u_n)_{n\geq 1} \in H^1_V(\mathbb{R}^N)$ is bounded. From Lemma 1.2, the embedding $H^1_V(\mathbb{R}^N) \hookrightarrow L_r(\mathbb{R}^N)$ is compact for $2 \leq r < 2^*$. Therefore, $u \not\equiv 0$. Using this embedding and lower semicontinuity, we conclude that $u_n \to u_a$ strongly in $H^1_V(\mathbb{R}^N)$. Continuing, we use relation (22) and we get

$$\lambda_n = \frac{\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla v\, dx + \int_{\mathbb{R}^N} V(x) u_n \cdot v\, dx - \beta \int_{\mathbb{R}^N} |u_n|^{p-2}u_n v\, dx - \int_{\mathbb{R}^N} |u_n|^{q-2} u_n v\, dx}{\int_{\mathbb{R}^N} u_n \cdot v\, dx} + o_n(1).$$

Testing the above equality with $v = u_n$, taking $n \to +\infty$ and applying the strong convergence, we can find $\lambda_a \in \mathbb{R}$ such that

$$\lambda_a = \frac{\int_{\mathbb{R}^N} |\nabla u_a|^2\, dx + \int_{\mathbb{R}^N} V(x)|u_a|^2\, dx - \beta \int_{\mathbb{R}^N} |u_a|^p\, dx - \int_{\mathbb{R}^N} |u_a|^q\, dx}{a}. \qquad (23)$$

Together (u_a, λ_a) solve PDE (1). □

Lemma 4.2. *The mapping $a \mapsto e(a)$ is locally Lipschitz on $(0, +\infty)$ and $a \mapsto \lambda_a$ is bounded on any compact subsets of $(0, +\infty)$.*

Proof. Let $I = [a_1, a_2] \subset (0, \infty)$ and let $u_a \in S(a)$ be a minimizer of $e(a)$. For $b \in I$, set $s = \sqrt{b/a}$. Then $\|su_a\|_2^2 = b$, hence $su_a \in S(b)$ and

$$e(b) \leq E(su_a).$$

A direct computation gives

$$E(su_a) - E(u_a) = \frac{s^2-1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_a|^2 + V|u_a|^2\right) - \frac{\beta(s^p-1)}{p}\int_{\mathbb{R}^N} |u_a|^p - \frac{s^q-1}{q}\int_{\mathbb{R}^N} |u_a|^q.$$

By Lemma 4.1, there exists $L_I > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla u_a|^2 + V|u_a|^2\right) + \beta \int_{\mathbb{R}^N} |u_a|^p + \int_{\mathbb{R}^N} |u_a|^q \le L_I \quad \text{for all } a \in I.$$

Hence

$$e(b) - e(a) \le L_I\Big(|s^2-1| + |s^p - 1| + |s^q - 1|\Big).$$

Since $a, b \in I$, the parameter s ranges in a compact interval, and the maps $t \mapsto t^k$ $(k = 2, p, q)$ are Lipschitz there. Thus $|s^k - 1| \le C_I |s-1|$ for some $C_I > 0$. Moreover,

$$|s-1| = \left|\sqrt{\frac{b}{a}} - 1\right| = \frac{|b-a|}{\sqrt{a}(\sqrt{b}+\sqrt{a})} \le \frac{|b-a|}{2a_1}.$$

Combining the above estimates yields

$$|e(b) - e(a)| \le C_I |b-a| \qquad \text{for all } a, b \in I,$$

so e is Lipschitz continuous on I. The boundedness of λ_a follows from (23). $\square$

Small-mass regime $a \to 0$

In the small-mass regime, the nonlinear ground state is selected by the linear ground state, and its maximum localizes at the unique maximum point of the first eigenfunction.

Lemma 4.3. *Let $H = -\Delta + V$ on $\mathbb{R}^N$, where V is such that H has compact resolvent (in particular, the spectrum is discrete). Let ω_1 be the first eigenvalue of H, assumed simple, with positive L^2–normalized eigenfunction ϕ_1:*

$$H\phi_1 = \omega_1\phi_1, \qquad \|\phi_1\|_2 = 1, \qquad \phi_1 > 0.$$

Fix exponents $2 < q < p < 2^*$ and $\beta > 0$. For each $a > 0$, let (u_a, λ_a) solve

$$Hu_a = \lambda_a u_a + \beta|u_a|^{p-2}u_a + |u_a|^{q-2}u_a, \qquad \|u_a\|_2 = a. \tag{24}$$

Then, as $a \to 0$,

$$u_a = a\phi_1 + o(a) \quad \text{in } H^1(\mathbb{R}^N),$$

and

$$\lambda_a = \omega_1 - \beta c_p a^{p-2} - c_q a^{q-2} + o(a^{p-2}), \qquad c_p := \int_{\mathbb{R}^N} \phi_1^p, \quad c_q := \int_{\mathbb{R}^N} \phi_1^q.$$

Proof. Let P denote the orthogonal projection onto $\operatorname{span}\{\phi_1\}$ in L^2 and set

$$u_a = a\phi_1 + w_a, \qquad \langle w_a, \phi_1\rangle_{L^2} = 0,$$

and $\lambda_a = \omega_1 - \mu_a$. Substituting into the equation gives

$$(H - \omega_1)w_a + \mu_a(a\phi_1 + w_a) = F_p(a\phi_1 + w_a) + \beta F_q(a\phi_1 + w_a),$$

where $F_r(u) = |u|^{r-2}u$. Since ω_1 is simple, the spectral gap $\omega_2 - \omega_1 > 0$ implies that $(H - \omega_1 + \mu_a)$ is coercive on $\{\phi_1\}^\perp$ for a small. Testing against w_a and using Sobolev embeddings, we obtain

$$\|w_a\|_{H^1} \lesssim \|a\phi_1 + w_a\|_{H^1}^{p-1} + \|a\phi_1 + w_a\|_{H^1}^{q-1}.$$

A standard bootstrap argument then yields

$$\|w_a\|_{H^1} = O(a^{q-1}) = o(a),$$

since $q > 2$. Hence

$$u_a = a\phi_1 + o(a) \quad \text{in } H^1(\mathbb{R}^N).$$

Projecting onto ϕ_1 and expanding the nonlinear terms gives

$$\mu_a a = \beta a^{p-1} \int\limits_{\mathbb{R}^N} \phi_1^p + a^{q-1} \int\limits_{\mathbb{R}^N} \phi_1^q + o(a^{p-1}),$$

which yields

$$\lambda_a = \omega_1 - \beta \left(\int\limits_{\mathbb{R}^N} \phi_1^p \right) a^{p-2} - \left(\int\limits_{\mathbb{R}^N} \phi_1^q \right) a^{q-2} + o(a^{p-2}). \quad \square$$

Lemma 4.4. *Let the first eigenfunction ϕ_1 have a unique global maximum at $x_1 \in \mathbb{R}^N$. Let x_a be a maximum point of u_a. Then*

$$x_a \to x_1 \quad \text{as } a \to 0.$$

Proof. Define $v_a := u_a/a$. By Lemma 4.3,

$$v_a \to \phi_1 \quad \text{in } H^1(\mathbb{R}^N).$$

Standard elliptic regularity applied to the equation satisfied by v_a implies that this convergence is uniform on $\mathbb{R}^N$.

Now, fix $\varepsilon > 0$. Since ϕ_1 has a unique global maximum at x_1, there exists $\delta > 0$ such that

$$\phi_1(x_1) > \sup_{|x-x_1| \geq \varepsilon} \phi_1(x) + 3\delta.$$

By uniform convergence, for a sufficiently small,

$$|v_a(x) - \phi_1(x)| \leq \delta \quad \text{for all } x \in \mathbb{R}^N.$$

Hence, if $|x - x_1| \geq \varepsilon$,

$$v_a(x) \leq \phi_1(x) + \delta \leq \phi_1(x_1) - 2\delta < \phi_1(x_1) - \delta \leq v_a(x_1).$$

Multiplying by a, we obtain

$$u_a(x) < u_a(x_1) \quad \text{for all } |x - x_1| \geq \varepsilon.$$

Let x_a be a maximum point of u_a. If $|x_a - x_1| \geq \varepsilon$, then

$$u_a(x_a) \leq \sup_{|x-x_1|\geq\varepsilon} u_a(x) < u_a(x_1),$$

which contradicts the maximality of x_a. Therefore, for a sufficiently small,

$$x_a \in B_\varepsilon(x_1).$$

Since $\varepsilon > 0$ is arbitrary, this proves $x_a \to x_1$ as $a \to 0$. □

Proof of Theorem 2.1. The proof is a direct result of Lemmas 4.3 and 4.4. □

Large-mass regime $a \to +\infty$

Heuristic. Before analyzing the large-mass regime, we briefly explain the scaling intuition that underlies the blow-up analysis. For a large mass $a \to \infty$, the L^2 constraint forces the solution to develop a large amplitude. The competition between the two nonlinearities is resolved by comparing their scaling exponents. Suppose we rescale the solution as $u(x) = \mu^{\frac{1}{p-2}} w(\mu^{1/2}x)$ with $\mu = -\lambda \to \infty$. Under this p-scaling, the equation becomes

$$-\Delta w + \mu^{-1}V(\cdot)w = -w + \beta w^{p-1} + \mu^{-\delta} w^{q-1}, \qquad \delta = \frac{p-q}{p-2} > 0.$$

For $\mu \to \infty$, the potential term vanishes (since $V \geq 0$ and we concentrate near a well where $V = 0$), and the q-nonlinearity is suppressed by the factor $\mu^{-\delta} \to 0$. Thus the leading-order balance is between $-\Delta w$, $-w$, and βw^{p-1}, yielding convergence to $\beta^{-1/(p-2)}Q_p$. Conversely, if we use the q-scaling $u(x) = \mu^{\frac{1}{q-2}} z(\mu^{1/2}x)$, the p-term carries a factor $\mu^{-\gamma} \to 0$ with $\gamma = (p-q)/(q-2) > 0$, so the q-nonlinearity dominates for small mass. The transition between these regimes occurs when the suppressed nonlinearity becomes comparable to the dominant one, leading to the crossover scale $a_\times(\beta) \asymp \beta^{-\frac{4-N(q-2)}{2(p-q)}}$.

Lemma 4.5. *Let $\mu_a = -\lambda_a$, where λ_a is the Lagrange multiplier for the minimizer u_a for $e(a)$. Then* $\lim_{a\to+\infty} \mu_a = +\infty$, *provided that*

$$\beta > \frac{p\gamma_p \|Q_p\|_2^{p-2}}{2}.$$

Proof. From the Euler–Lagrange equation satisfied by u_a, multiplying by u_a and integrating over $\mathbb{R}^N$, we obtain

$$\lambda_a a = 2e(a) + \beta\Big(\frac{2}{p} - 1\Big)\|u_a\|_p^p + \Big(\frac{2}{q} - 1\Big)\|u_a\|_q^q. \tag{25}$$

Since $p > 2$ and $q > 2$, the coefficients $\frac{2}{p} - 1$ and $\frac{2}{q} - 1$ are negative. Hence,

$$\lambda_a a \leq 2e(a). \tag{26}$$

Therefore, it suffices to show that

$$\frac{e(a)}{a} \longrightarrow -\infty \quad \text{as } a \to +\infty. \tag{27}$$

Let $x_j \in \mathbb{R}^N$ be such that $V(x_j) = 0$, and let Q_p be the positive ground state solution of (6). For $\varepsilon > 0$, define

$$u_\varepsilon(x) := \varepsilon^{-N/2} Q_p\left(\frac{x - x_0}{\varepsilon}\right).$$

Then

$$\|u_\varepsilon\|_2^2 = \|Q_p\|_2^2 = m_p, \quad \|\nabla u_\varepsilon\|_2^2 = \varepsilon^{-2}\|\nabla Q_p\|_2^2,$$
$$\|u_\varepsilon\|_p^p = \varepsilon^{-p\gamma_p}\|Q_p\|_p^p, \quad \|u_\varepsilon\|_q^q = \varepsilon^{-q\gamma_q}\|Q_p\|_q^q,$$

and, since $V(x_j) = 0$ and V is continuous,

$$\int\limits_{\mathbb{R}^N} V(x)|u_\varepsilon|^2\, dx = o(1) \quad \text{as } \varepsilon \to 0.$$

Set $t := \sqrt{\dfrac{a}{m_p}}$ and define $v_{a,\varepsilon} := t u_\varepsilon$. Then $\|v_{a,\varepsilon}\|_2^2 = a$, so $v_{a,\varepsilon} \in S(a)$ and

$$e(a) \leq E(v_{a,\varepsilon}).$$

A direct computation yields

$$\begin{aligned} E(v_{a,\varepsilon}) &< \frac{1}{2}\frac{\|\nabla Q_p\|_2^2}{\|Q_p\|_2^2} a\varepsilon^{-2} - \frac{\beta}{p}\frac{\|Q_p\|_p^p}{\|Q_p\|_2^p} a^{p/2}\varepsilon^{-p\gamma_p} + o(a) \\ &= \frac{\gamma_p}{2(1-\gamma_p)} a\varepsilon^{-2} - \frac{\beta}{p(1-\gamma_p)\|Q_p\|_2^{p-2}} a^{p/2}\varepsilon^{-p\gamma_p} + o(a). \end{aligned} \tag{28}$$

Choose $\varepsilon = a^{-k}$ with

$$0 < k = \frac{p-2}{2(2 - p\gamma_p)},$$

or equivalently $\frac{p}{2} + kp\gamma_p = 1 + 2k$ and this is well defined since $p < 2 + \frac{4}{N}$. With this choice, both the gradient term and the p-nonlinearity scale like a^σ, where

$$\sigma := \frac{p}{2} + kp\gamma_p = 1 + 2k > 1.$$

Putting these into (28), we obtain

$$E(v_{a,\varepsilon}) < a^\sigma\left(\frac{\gamma_p}{2(1-\gamma_p)} - \frac{\beta}{p(1-\gamma_p)\|Q_p\|_2^{p-2}}\right) + o(a).$$

Let $\beta > 0$ be chosen such that

$$\beta > \frac{p\gamma_p\|Q_p\|_2^{p-2}}{2}.$$

Hence, for a sufficiently large,

$$e(a) \leq -c\, a^\sigma \quad \text{for some } c > 0.$$

Dividing by a gives

$$\frac{e(a)}{a} \leq -c\, a^{\sigma-1} \longrightarrow -\infty \quad \text{as } a \to +\infty,$$

which proves (27). Finally, combining with (26), we conclude that

$$\lambda_a \longrightarrow -\infty.$$

Equivalently, $\lim_{a\to+\infty} \mu_a = +\infty$. This completes the proof. □

Remark 4.6. Let

$$\beta_* := \frac{p\gamma_p \|Q_p\|_2^{p-2}}{2}.$$

The condition $\beta > \beta_*$ appearing in Theorem 2.2 and Lemma 4.5 is sharp in the following sense:

- **(i) If $\beta > \beta_*$:** The energy functional becomes negative for sufficiently large mass when tested against p-core bubbles. Consequently, $\mu_a \to \infty$ and the rescaled solution converges to $\beta^{-1/(p-2)} Q_p$. The p-nonlinearity is responsible for the collapse.
- **(ii) If $\beta = \beta_*$:** The leading-order coefficient in the p-bubble energy expansion vanishes. Higher-order terms (from the potential or the q-nonlinearity) determine the sign. In this borderline case, the asymptotic behavior is expected to be more delicate, possibly involving logarithmic corrections or a different blow-up rate.
- **(iii) If $\beta < \beta_*$:** The p-bubble energy is non-negative to leading order, so the p-nonlinearity alone cannot drive the energy to $-\infty$ as $a \to \infty$. In this regime, the q-nonlinearity (which is always mass-subcritical) prevents collapse, and the large-mass behavior is governed by the q-core dynamics described in Theorem 2.3(i), provided the mass remains below the crossover scale. For sufficiently large mass when β is small, the competition between the two nonlinearities may produce a different type of ground state, possibly with a more complex structure (e.g., two-bump solutions).

Thus β_* is the exact threshold separating the regime where pure p-core blow-up occurs from regimes where it does not.

Therefore, from now on, we always assume $\beta > \frac{p\gamma_p\|Q_p\|_2^{p-2}}{2}$. Now, recall from Lemma 4.5 that

$$\lambda_a a \leq e(a) \leq E(v_{a,\varepsilon}) \leq -c\, a^{\sigma},$$

for some $c > 0$ and large $a > 0$. From this, we conclude

$$\mu_a \geq c\, a^{\sigma-1},$$

where

$$\sigma = \frac{p - p\gamma_p}{2 - p\gamma_p} > 1.$$

We will use this quantitative growth rate for the next lemma.

Lemma 4.7. *Let $\beta > \frac{p\gamma_p\|Q_p\|_2^{p-2}}{2}$. Let $\mu_a = -\lambda_a$, where λ_a is the Lagrange multiplier for the minimizer u_a for $e(a)$, and $\varepsilon_a = \mu_a^{-1/2}$, and consider the scaling*

$$w_a(x) = \mu_a^{-\frac{1}{p-2}} u_a(x_a + \varepsilon_a x),$$

where $x_a \in \mathbb{R}^N$ *is a maximum point. Then, after passing to a subsequence, we have the strong convergence* $w_a \to \beta^{-\frac{1}{p-2}} Q_p$ *in* $H^1(\mathbb{R}^N)$*. Also,*

$$\mu_a = \left(\frac{\beta^{\frac{2}{p-2}}}{m_p} a \right)^{\frac{p-2}{2-p\gamma_p}} (1 + o(1)), \qquad a \to +\infty.$$

Proof. From the definition of w_a,

$$\|w_a\|_2^2 = a\, \mu_a^{\frac{N}{2} - \frac{2}{p-2}}.$$

Using the growth of μ_a (Remark 4.6), $\|w_a\|_2$ is uniformly bounded. Testing the rescaled equation

$$-\Delta w_a + \mu_a^{-1} V(x_a + \varepsilon_a x) w_a = -w_a + \beta w_a^{p-1} + \mu_a^{-\delta} w_a^{q-1}$$

with w_a and applying the Gagliardo–Nirenberg inequality, we obtain

$$\|\nabla w_a\|_2^2 \lesssim \|\nabla w_a\|_2^{p\gamma_p} + \|\nabla w_a\|_2^{q\gamma_q}.$$

Since $0 < q\gamma_q < p\gamma_p < 2$, $\{w_a\}$ is bounded in $H^1(\mathbb{R}^N)$. Up to a subsequence,

$$w_a \rightharpoonup w \quad \text{in } H^1(\mathbb{R}^N).$$

Because x_a is a maximum point of u_a, 0 is a maximum of w_a, hence $w_a(0) \geq c > 0$ for large a. Passing to the limit gives $w \not\equiv 0$. Letting $a \to \infty$ in the rescaled equation and using $\mu_a^{-1}, \mu_a^{-\delta} \to 0$, we obtain

$$-\Delta w + w = \beta w^{p-1} \quad \text{in } \mathbb{R}^N.$$

By the classification of positive solutions,

$$w = \beta^{-\frac{1}{p-2}} Q_p(\cdot - y).$$

Define the autonomous functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p.$$

Then $J'(w_a) \to 0$ in H^{-1}, so $\{w_a\}$ is a bounded Palais–Smale sequence. Moreover,

$$J(w_a) = \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} w_a^p + o(1) \to J(Q_p) = m.$$

By concentration–compactness, vanishing and dichotomy are excluded, hence $w_a \to w$ strongly in $H^1(\mathbb{R}^N)$. Since 0 remains a maximum point, the translation parameter vanishes and

$$w_a \to \beta^{-\frac{1}{p-2}} Q_p \quad \text{strongly in } H^1(\mathbb{R}^N).$$

Using again

$$\|w_a\|_2^2 = a\,\mu_a^{\frac{N}{2}-\frac{2}{p-2}},$$

and the strong convergence,

$$a\,\mu_a^{\frac{N}{2}-\frac{2}{p-2}} = \beta^{-\frac{2}{p-2}} m_p(1+o(1)).$$

Since $\frac{N}{2} - \frac{2}{p-2} = -\frac{2-p\gamma_p}{p-2}$, we obtain

$$\mu_a^{\frac{2-p\gamma_p}{p-2}} = \frac{\beta^{\frac{2}{p-2}}}{m_p}\,a(1+o(1)),$$

that is,

$$\mu_a = \left(\frac{\beta^{\frac{2}{p-2}}}{m_p}\,a\right)^{\frac{p-2}{2-p\gamma_p}}(1+o(1)), \qquad a \to \infty. \quad \square$$

Lemma 4.8. *Let u_a be a global minimizer of $e(a)$, let λ_a be the associated Lagrange multiplier, and set*

$$\mu_a := -\lambda_a, \qquad \varepsilon_a := \mu_a^{-1/2}.$$

Let x_a be a maximum point of u_a. Then

$$\operatorname{dist}(x_a, \mathcal{Z}) \to 0 \qquad \text{as } a \to +\infty,$$

where $\mathcal{Z} = \{x \in \mathbb{R}^N : V(x) = 0\}$.

Proof. We argue by contradiction. Suppose that the conclusion fails. Then there exists a $d_0 > 0$ and a sequence $a_n \to +\infty$ such that

$$\operatorname{dist}(x_{a_n}, \mathcal{Z}) \geq d_0 \qquad \text{for all } n.$$

Since $\mathcal{Z}$ is finite and $V \in C(\mathbb{R}^N)$ with $V \geq 0$ and isolated zeros, there exists

$$\alpha_0 := \min\left\{V(x) : \operatorname{dist}(x, \mathcal{Z}) \geq d_0/2\right\} > 0.$$

Define the rescaled functions

$$w_a(x) := \mu_a^{-\frac{1}{p-2}}\,u_a(x_a + \varepsilon_a x).$$

By Lemma 4.7, we have

$$w_a \to \beta^{-\frac{1}{p-2}} Q_p \quad \text{strongly in } H^1(\mathbb{R}^N),$$

and hence strongly in $L^2(\mathbb{R}^N)$. Therefore, for any fixed $R > 0$,

$$\int_{B_R} |w_a|^2\,dx \longrightarrow \int_{B_R} \beta^{-\frac{2}{p-2}} Q_p^2\,dx.$$

Choose $R > 0$ large enough so that

$$\int_{B_R} \beta^{-\frac{2}{p-2}} Q_p^2 \, dx \geq \frac{2}{3} \int_{\mathbb{R}^N} \beta^{-\frac{2}{p-2}} Q_p^2 \, dx.$$

Then, for all sufficiently large a,

$$\int_{B_R} |w_a|^2 \, dx \geq \frac{2}{3} \int_{\mathbb{R}^N} |w_a|^2 \, dx.$$

Using the change of variables $y = x_a + \varepsilon_a x$ and the identity

$$\int_{\mathbb{R}^N} |u_a(y)|^2 \, dy = \mu_a^{\frac{2}{p-2}} \varepsilon_a^N \int_{\mathbb{R}^N} |w_a(x)|^2 \, dx = a,$$

we obtain

$$\int_{B_{R\varepsilon_a}(x_a)} |u_a(y)|^2 \, dy = \mu_a^{\frac{2}{p-2}} \varepsilon_a^N \int_{B_R} |w_a(x)|^2 \, dx \geq \frac{2}{3}\, a$$

for all sufficiently large a. Since $\operatorname{dist}(x_{a_n}, \mathcal{Z}) \geq d_0$ for all n, for any $y \in B_{R\varepsilon_{a_n}}(x_{a_n})$ we have

$$\operatorname{dist}(y, \mathcal{Z}) \geq \operatorname{dist}(x_{a_n}, \mathcal{Z}) - |y - x_{a_n}| \geq d_0 - R\varepsilon_{a_n}.$$

Since $\varepsilon_{a_n} \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $R\varepsilon_{a_n} \leq d_0/2$ for all $n \geq n_0$. Hence, for all $n \geq n_0$,

$$\operatorname{dist}(y, \mathcal{Z}) \geq \frac{d_0}{2} \qquad \text{for all } y \in B_{R\varepsilon_{a_n}}(x_{a_n}),$$

which implies

$$B_{R\varepsilon_{a_n}}(x_{a_n}) \subset \big\{x \in \mathbb{R}^N : \operatorname{dist}(x, \mathcal{Z}) \geq d_0/2\big\}.$$

Hence, $V(x) \geq \alpha_0$ on $B_{R\varepsilon_{a_n}}(x_{a_n})$, and therefore

$$\int_{\mathbb{R}^N} V(x)|u_{a_n}(x)|^2 \, dx \geq \int_{B_{R\varepsilon_{a_n}}(x_{a_n})} V(x)|u_{a_n}(x)|^2 \, dx \geq \alpha_0 \int_{B_{R\varepsilon_{a_n}}(x_{a_n})} |u_{a_n}(x)|^2 \, dx \geq \frac{2}{3}\, \alpha_0 \, a_n.$$

Define the zero-potential constrained infimum

$$e_0(a) := \inf\Big\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\beta}{p}\int_{\mathbb{R}^N} |u|^p - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q : \; u \in H^1(\mathbb{R}^N), \; \|u\|_2^2 = a\Big\}.$$

From the previous estimate and the definition of E, we obtain

$$e(a_n) = E(u_{a_n}) \geq e_0(a_n) + \frac{1}{3}\, \alpha_0 \, a_n. \tag{29}$$

On the other hand, by testing the energy with standard concentrating functions supported near any point $x_j \in \mathcal{Z}$ (where $V(x_j) = 0$), one has

$$e(a) \leq e_0(a) + o(a) \qquad \text{as } a \to +\infty. \tag{30}$$

Combining (29) and (30) yields

$$e_0(a_n) + \frac{1}{3}\,\alpha_0\, a_n \le e(a_n) \le e_0(a_n) + o(a_n),$$

which is impossible since $\alpha_0 > 0$ and $a_n \to +\infty$. Therefore,

$$\operatorname{dist}(x_a, \mathcal{Z}) \to 0 \qquad \text{as } a \to +\infty.$$

This completes the proof. □

Lemma 4.9. *Assume* $(V1) - (V3)$ *and* $2 < q < p < 2 + \frac{4}{N}$, $\beta > 0$. *Let* u_a *be a global minimizer of* $e(a)$ *on* $S(a)$ *with associated Lagrange multiplier* λ_a. *Let* x_a *be a maximum point of* u_a *and define*

$$w_a(x) := \mu_a^{-\frac{1}{p-2}} u_a(x_a + \varepsilon_a x).$$

Assume Lemma 4.7 holds, i.e., (up to a subsequence)

$$w_a \to w := \beta^{-\frac{1}{p-2}} Q_p \quad \text{strongly in } H^1(\mathbb{R}^N),$$

where Q_p *is the positive ground state of* (6). *For each* $x_j \in \mathcal{Z}$ *let*

$$V(x) = c_j |x - x_j|^{\ell_j} + o(|x - x_j|^{\ell_j}) \quad \text{as } x \to x_j,$$

with $c_j > 0$ *and* $\ell_j \ge 2$, *and define*

$$\mathcal{M}_\ell := \int_{\mathbb{R}^N} |x|^\ell w(x)^2\, dx = \beta^{-\frac{2}{p-2}} \int_{\mathbb{R}^N} |x|^\ell Q_p(x)^2\, dx \in (0, \infty).$$

Then for every $j \in \{1, \ldots, m\}$ *there exists* $U_{a,j} \in S(a)$ *such that*

$$E(U_{a,j}) \le E_0(u_a) + \frac{1}{2}\, c_j\, \mathcal{M}_{\ell_j}\, \varepsilon_a^{\ell_j} + o(\varepsilon_a^{\ell_j}) \qquad (a \to +\infty). \tag{31}$$

If, up to a subsequence, $x_a \to x_{j_\infty} \in \mathcal{Z}$, *then*

$$E(u_a) \ge E_0(u_a) + \frac{1}{2}\, c_{j_\infty}\, \mathcal{M}_{\ell_{j_\infty}}\, \varepsilon_a^{\ell_{j_\infty}} + o(\varepsilon_a^{\ell_{j_\infty}}) \qquad (a \to +\infty). \tag{32}$$

Consequently, for every $k \in \{1, \ldots, m\}$,

$$c_{j_\infty}\, \mathcal{M}_{\ell_{j_\infty}}\, \varepsilon_a^{\ell_{j_\infty}} \le c_k\, \mathcal{M}_{\ell_k}\, \varepsilon_a^{\ell_k} + o(\varepsilon_a^{\ell_{j_\infty}}) + o(\varepsilon_a^{\ell_k}). \tag{33}$$

Proof. We write

$$E(u) = E_0(u) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx, \qquad E_0(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\beta}{p}\int_{\mathbb{R}^N} |u|^p - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q.$$

Fix j. Choose $\chi \in C_c^\infty(\mathbb{R}^N)$ radial such that $\chi \equiv 1$ on B_1, $\chi \equiv 0$ on $\mathbb{R}^N \setminus B_2$, $0 \le \chi \le 1$. Fix $R > 1$ and define the cut-off bubble

$$\widetilde{U}_{a,j}(x) := \mu_a^{\frac{1}{p-2}}\, w\Big(\frac{x - x_j}{\varepsilon_a}\Big)\, \chi\Big(\frac{x - x_j}{R\varepsilon_a}\Big).$$

Let $t_a > 0$ be such that $U_{a,j} := t_a \widetilde{U}_{a,j} \in S(a)$. By the change of variables $x = x_j + \varepsilon_a y$,

$$\|\widetilde{U}_{a,j}\|_2^2 = \mu_a^{\frac{2}{p-2}} \varepsilon_a^N \int_{\mathbb{R}^N} w(y)^2 \chi(y/R)^2 \, dy = \mu_a^{\frac{2}{p-2}} \varepsilon_a^N \Big(\int_{\mathbb{R}^N} w^2 + o_R(1) \Big),$$

hence $t_a = 1 + o_R(1)$ as $a \to \infty$, where $o_R(1) \to 0$ as $R \to \infty$. Moreover, since E_0 is translation invariant and $\chi(\cdot/R) \to 1$,

$$E_0(U_{a,j}) = E_0\Big(\mu_a^{\frac{1}{p-2}} w(\cdot/\varepsilon_a)\Big) + o_R(1).$$

Using $E(u_a) = e(a) \le E(U_{a,j})$ and $E_0(u_a) \le E_0(U_{a,j}) + o_R(1)$, we can write (absorbing $o_R(1)$ into the final error term)

$$E(U_{a,j}) \le E_0(u_a) + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |U_{a,j}(x)|^2 \, dx + o_R(1).$$

Now compute the potential term. On $\operatorname{supp}\chi(\frac{x-x_j}{R\varepsilon_a})$ we have $|x - x_j| \le 2R\varepsilon_a$, hence

$$V(x_j + \varepsilon_a y) = c_j \varepsilon_a^{\ell_j} |y|^{\ell_j} + o(\varepsilon_a^{\ell_j}) \quad \text{uniformly for } |y| \le 2R.$$

Since $\chi(\cdot/R) \to 1$ and $t_a = 1 + o_R(1)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x)|U_{a,j}(x)|^2 \, dx &= t_a^2 \mu_a^{\frac{2}{p-2}} \varepsilon_a^N \int_{\mathbb{R}^N} V(x_j + \varepsilon_a y) \, w(y)^2 \chi(y/R)^2 \, dy \\
&= t_a^2 \mu_a^{\frac{2}{p-2}} \varepsilon_a^N \Big(c_j \varepsilon_a^{\ell_j} \int_{\mathbb{R}^N} |y|^{\ell_j} w(y)^2 \chi(y/R)^2 \, dy + o(\varepsilon_a^{\ell_j}) \Big) \\
&= t_a^2 \mu_a^{\frac{2}{p-2} - \frac{N}{2}} \Big(c_j \varepsilon_a^{\ell_j} \int_{\mathbb{R}^N} |y|^{\ell_j} w(y)^2 \chi(y/R)^2 \, dy + o(\varepsilon_a^{\ell_j}) \Big) \\
&= t_a^2 \frac{m_p}{a} \beta^{-\frac{2}{p-2}} \Big(c_j \varepsilon_a^{\ell_j} \int_{\mathbb{R}^N} |y|^{\ell_j} w(y)^2 \chi(y/R)^2 \, dy + o(\varepsilon_a^{\ell_j}) \Big) \\
&= c_j \varepsilon_a^{\ell_j} \Big(\mathcal{M}_{\ell_j} + o(\varepsilon_a^{\ell_j}) \Big)(1 + o_R(1)).
\end{aligned}$$

Letting first $a \to \infty$ and then $R \to \infty$ yields (31). Now, we assume $x_a \to x_{j_\infty}$. Using the blow-up variables $x = x_a + \varepsilon_a y$,

$$\int_{\mathbb{R}^N} V(x) |u_a(x)|^2 \, dx = \mu_a^{\frac{2}{p-2}} \varepsilon_a^N \int_{\mathbb{R}^N} V(x_a + \varepsilon_a y) \, |w_a(y)|^2 \, dy.$$

Fix $R > 0$ and split the integral into B_R and $\mathbb{R}^N \setminus B_R$. By $w_a \to w$ in L^2 and $w \in L^2$, choose R large such that $\int_{\mathbb{R}^N \setminus B_R} w^2$ is arbitrarily small; then for a large $\int_{\mathbb{R}^N \setminus B_R} |w_a|^2$ is also small. Using the flatness expansion at x_{j_∞} and $x_a \to x_{j_\infty}$, uniformly for $|y| \le R$,

$$V(x_a + \varepsilon_a y) = c_{j_\infty} \varepsilon_a^{\ell_{j_\infty}} |y|^{\ell_{j_\infty}} + o(\varepsilon_a^{\ell_{j_\infty}}).$$

Hence,

$$\int_{B_R} V(x_a+\varepsilon_a y)|w_a(y)|^2\,dy = c_{j_\infty}\varepsilon_a^{\ell_{j_\infty}} \int_{B_R} |y|^{\ell_{j_\infty}} w(y)^2\,dy + o(\varepsilon_a^{\ell_{j_\infty}}),$$

where we used $w_a \to w$ in $L^2(B_R)$. Letting $R \to \infty$ gives

$$\int_{\mathbb{R}^N} V(x_a+\varepsilon_a y)|w_a(y)|^2\,dy \geq c_{j_\infty}\varepsilon_a^{\ell_{j_\infty}} \mathcal{M}_{\ell_{j_\infty}} + o(\varepsilon_a^{\ell_{j_\infty}}).$$

Therefore,

$$\int_{\mathbb{R}^N} V|u_a|^2 \geq c_{j_\infty}\mathcal{M}_{\ell_{j_\infty}}\varepsilon_a^{\ell_{j_\infty}} + o(\varepsilon_a^{\ell_{j_\infty}}),$$

and inserting this into $E(u_a) = E_0(u_a) + \frac{1}{2}\int_{\mathbb{R}^N} V|u_a|^2$ yields (32). Finally, combining (31) (with k in place of j) and (32) gives (33). □

Remark 4.10. The functional E_0 is translation invariant: for every $u \in H^1(\mathbb{R}^N)$ and every $y \in \mathbb{R}^N$, setting $u_y(x) := u(x-y)$ one has

$$\int_{\mathbb{R}^N} |\nabla u_y|^2 = \int_{\mathbb{R}^N} |\nabla u|^2, \qquad \int_{\mathbb{R}^N} |u_y|^r = \int_{\mathbb{R}^N} |u|^r \quad (r=p,q),$$

hence $E_0(u_y) = E_0(u)$. Therefore, the leading-order core energy of a single concentrating bubble is independent of the well index, and the well selection is entirely produced by the potential term $\frac{1}{2}\int V|u|^2$. The cut-off $\chi(\cdot/(R\varepsilon_a))$ is introduced only to localize the test functions where the expansion of V is valid; its contribution to E_0 is supported in an annulus where the bubble is exponentially small, and vanishes as $R \to \infty$.

Lemma 4.11. *Let u_a be a global minimizer of $e(a)$ and let x_a be a maximum point of u_a. Assume that, up to a subsequence,*

$$x_a \to x_{j_\infty} \in \mathcal{Z} \qquad \text{as } a \to +\infty.$$

Then the following conclusions hold:

(i) $\ell_{j_\infty} = \max_{1\leq j\leq m} \ell_j =: \ell_*$;
(ii) $j_\infty \in \arg\min\{c_j : \ \ell_j = \ell_*\}$.

In particular, if the minimizer of c_j over $\{j : \ell_j = \ell_*\}$ is unique, say j_*, then

$$x_a \to x_{j_*} \qquad \text{as } a \to +\infty.$$

Proof. Fix $k \in \{1,\dots,m\}$. By the upper bound obtained from the test function concentrated near x_k and the lower bound for the true minimizer u_a (see Lemmas 4.9), we have

$$c_{j_\infty}\mathcal{M}_{\ell_{j_\infty}} \leq c_k\,\mathcal{M}_{\ell_k}\,\varepsilon_a^{\ell_k-\ell_{j_\infty}} + o(1) + o\left(\varepsilon_a^{\ell_k-\ell_{j_\infty}}\right), \qquad a \to +\infty. \tag{34}$$

Assume by contradiction that there exists an index k such that $\ell_k > \ell_{j_\infty}$. Then $\ell_k - \ell_{j_\infty} > 0$, and since $\varepsilon_a \to 0$ we have

$$\varepsilon_a^{\ell_k-\ell_{j_\infty}} \to 0, \qquad o\left(\varepsilon_a^{\ell_k-\ell_{j_\infty}}\right) \to 0.$$

Letting $a \to +\infty$ in (34) gives

$$c_{j_\infty}\,\mathcal{M}_{\ell_{j_\infty}} \leq 0,$$

which is impossible because $c_{j_\infty} > 0$ and $\mathcal{M}_{\ell_{j_\infty}} > 0$. Hence, no such k can exist, and therefore

$$\ell_{j_\infty} = \max_{1\leq j\leq m} \ell_j =: \ell_*.$$

Let k be such that $\ell_k = \ell_{j_\infty} = \ell_*$. Then $\ell_k - \ell_{j_\infty} = 0$, and (34) reduces to

$$c_{j_\infty}\,\mathcal{M}_{\ell_*} \leq c_k\,\mathcal{M}_{\ell_*} + o(1), \qquad a \to +\infty.$$

Letting $a \to +\infty$ and dividing by $\mathcal{M}_{\ell_*} > 0$, we obtain $c_{j_\infty} \leq c_k$. Since k was arbitrary among indices with $\ell_k = \ell_*$, this shows that

$$j_\infty \in \arg\min\{c_j :\ \ell_j = \ell_*\}.$$

If the minimizer is unique, say j_*, then any accumulation point of x_a must equal x_{j_*}, and therefore the full sequence converges:

$$x_a \to x_{j_*} \qquad \text{as } a \to +\infty.$$

This completes the proof. □

Proof of Theorem 2.2. (i) follows from Lemma 4.7, (ii) follows from Lemma 4.8, and (iii) follows from Lemma 4.11. □

Proof of Theorem 2.3. Let $u_a \in S(a)$ be a global minimizer of $e(a)$, with multiplier λ_a, and set $\mu_a := -\lambda_a$, $\varepsilon_a := \mu_a^{-1/2}$. Let x_a be a global maximum point of u_a. Then u_a solves

$$-\Delta u_a + V(x)u_a = \lambda_a u_a + \beta u_a^{p-1} + u_a^{q-1} \qquad \text{in } \mathbb{R}^N. \tag{35}$$

Define

$$z_a(x) := \mu_a^{-\frac{1}{q-2}} u_a(x_a + \varepsilon_a x), \qquad w_a(x) := \mu_a^{-\frac{1}{p-2}} u_a(x_a + \varepsilon_a x).$$

A direct computation yields

$$-\Delta z_a + \mu_a^{-1} V(x_a + \varepsilon_a x) z_a = -z_a + \beta\mu_a^{-\gamma} z_a^{p-1} + z_a^{q-1}, \tag{36}$$

$$-\Delta w_a + \mu_a^{-1} V(x_a + \varepsilon_a x) w_a = -w_a + \beta w_a^{p-1} + \mu_a^{-\delta} w_a^{q-1}, \tag{37}$$

where

$$\gamma = \frac{p-q}{q-2} > 0, \qquad \delta = \frac{p-q}{p-2} > 0.$$

Thus, in the q–scaling the p–term is weighted by $\beta\mu_a^{-\gamma}$, while in the p–scaling the q–term is weighted by $\mu_a^{-\delta}$. Let Q_q be the positive ground state of

$$-\Delta Q_q + Q_q = Q_q^{q-1} \quad \text{in } \mathbb{R}^N, \qquad m_q := \|Q_q\|_2^2.$$

Fix $x_j \in \mathcal{Z}$ and consider the standard bubble $\phi_\varepsilon(x) = \varepsilon^{-N/2} Q_q((x - x_j)/\varepsilon)$. Choosing $t^2 = a/m_q$ and using $V \geq 0$, one obtains

$$e(a) \leq E(t\phi_\varepsilon) \leq C_1\, a\, \varepsilon^{-2} - C_2\, a^{q/2} \varepsilon^{-\kappa_q}, \qquad \kappa_q = \frac{N(q-2)}{2},$$

where $C_1, C_2 > 0$ are positive constants. Balancing the two leading terms with $\varepsilon = a^{-k_q}$, $k_q = \frac{q-2}{4-N(q-2)}$, gives for a large

$$e(a) \leq -c_0\, a^{1+2k_q}. \tag{38}$$

On the other hand, multiplying (35) by u_a yields $E(u_a) \geq \frac{1}{2}\lambda_a a$, hence $\lambda_a a \leq 2e(a)$ and therefore

$$\mu_a = -\lambda_a \;\gtrsim\; \frac{-e(a)}{a} \;\gtrsim\; a^{2k_q}. \tag{39}$$

Consequently,

$$\beta \mu_a^{-\gamma} \lesssim \beta\, a^{-\frac{2(p-q)}{4-N(q-2)}}.$$

Thus the regime $\beta\mu_a^{-\gamma} \to 0$ holds provided

$$a \ll a_\times(\beta), \qquad a_\times(\beta) \asymp \beta^{\frac{4-N(q-2)}{2(p-q)}},$$

which proves (iii). Next, assume $a \to \infty$ with $a \ll a_\times(\beta)$, so that

$$\beta\mu_a^{-\gamma} \to 0. \tag{40}$$

Testing (36) by z_a and using (40) together with Gagliardo–Nirenberg inequalities yields a uniform bound $\|z_a\|_{H^1} \leq C$. Since 0 is a maximum point of z_a, evaluating (36) at 0 gives $z_a(0) \geq c_* > 0$. Hence, up to a subsequence, $z_a \rightharpoonup z$ in H^1 with $z \not\equiv 0$.

Moreover, since $\mu_a \to \infty$ and $\varepsilon_a \to 0$, we have $\mu_a^{-1} V(x_a + \varepsilon_a x) \to 0$ locally uniformly, and by (40) we may pass to the limit in (36) to obtain

$$-\Delta z + z = z^{q-1} \quad \text{in } \mathbb{R}^N.$$

Therefore, z is a positive solution, hence (by uniqueness up to translation) $z = Q_q(\cdot - x_0)$. Energy identities plus weak lower semicontinuity imply $\|z_a\|_{H^1} \to \|z\|_{H^1}$. Hence $z_a \to z$ strongly in $H^1(\mathbb{R}^N)$. Since z_a attains its maximum at 0, we have $x_0 = 0$, and thus

$$z_a \to Q_q \quad \text{strongly in } H^1(\mathbb{R}^N),$$

which is (i). Now, assume $a \to \infty$, $a \gg a_\times(\beta)$ and $\beta > p\gamma_p \|Q_p\|_2^{p-2}$. Then Theorem 2.2 applies and yields

$$w_a \to \beta^{-\frac{1}{p-2}} Q_p \quad \text{strongly in } H^1(\mathbb{R}^N),$$

which is (ii). Now, write $E(u) = E_0(u) + \frac{1}{2}\int_{\mathbb{R}^N} V|u|^2$ and recall that near each $x_j \in \mathcal{Z}$,

$$V(x_j + \varepsilon x) = c_j \varepsilon^{\ell_j} |x|^{\ell_j} + o(\varepsilon^{\ell_j}) \qquad (\varepsilon \to 0). \tag{41}$$

Assume $x_a \to x_{j_0} \in \mathcal{Z}$. Using the change of variables $x = x_a + \varepsilon_a y$,

$$\int\limits_{\mathbb{R}^N} V(x)|u_a(x)|^2\,dx = \mu_a^{\frac{2}{r-2}}\varepsilon_a^N \int\limits_{\mathbb{R}^N} V(x_a+\varepsilon_a y)\,U_a(y)^2\,dy,$$

where $(r, U_a) = (q, z_a)$ in the q–core regime and $(r, U_a) = (p, w_a)$ in the p–core regime. By (41),

$$V(x_a+\varepsilon_a y) = c_{j_0}\varepsilon_a^{\ell_{j_0}}|y|^{\ell_{j_0}} + o(\varepsilon_a^{\ell_{j_0}}) \quad \text{locally uniformly,}$$

hence

$$\int\limits_{\mathbb{R}^N} V|u_a|^2 = c_{j_0}\,\mu_a^{\frac{2}{r-2}}\varepsilon_a^{N+\ell_{j_0}} \int\limits_{\mathbb{R}^N} |y|^{\ell_{j_0}}U_a(y)^2\,dy + o\left(\mu_a^{\frac{2}{r-2}}\varepsilon_a^{N+\ell_{j_0}}\right).$$

Using the mass constraint $a = \mu_a^{\frac{2}{r-2}}\varepsilon_a^N\int_{\mathbb{R}^N} U_a^2$, we get

$$\int\limits_{\mathbb{R}^N} V|u_a|^2 = c_{j_0}\,a\,\varepsilon_a^{\ell_{j_0}}\frac{\int\limits_{\mathbb{R}^N} |y|^{\ell_{j_0}}U_a(y)^2\,dy}{\int\limits_{\mathbb{R}^N} U_a(y)^2\,dy} + o\left(a\varepsilon_a^{\ell_{j_0}}\right).$$

This completes the proof. □

CRediT authorship contribution statement

M.S.A., methodology and writing–original draft preparation. M.B.G., editing–original draft preparation. R.S., supervision and project administration. C.L., editing–original draft preparation.

Consent for publication

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References

[1] A. Ambrosetti, A. Malchiodi, S. Secchi, Multiplicity results for some nonlinear Schrödinger equations with potentials, Arch. Ration. Mech. Anal. 159 (3) (2001) 253–271.

[2] Thomas Bartsch, Sébastien de Valeriola, Normalized solutions of nonlinear Schrödinger equations, arXiv:1209.0950v1 [math.AP], 2012.

[3] Thomas Bartsch, Louis Jeanjean, Nicola Soave, Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$, J. Math. Pures Appl. (9) 106 (4) (2016) 583–614.

[4] J. Byeon, Z.Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 165 (2002) 295–316.

[5] J. Byeon, L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, Arch. Ration. Mech. Anal. 185 (2007) 185–200.

[6] Thierry Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematical Sciences/American Mathematical Society, New York/Providence, RI, 2003, xiv+323 pp.

[7] Manuel del Pino, Patricio L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differ. Equ. 4 (2) (1996) 121–137.

[8] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974) 324–353.

[9] Tianxiang Gou, Louis Jeanjean, Existence and orbital stability of standing waves for nonlinear Schrödinger systems, arXiv:1512.08952v2 [math.AP], 2016.

[10] Louis Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlinear Anal. 28 (10) (1997) 1633–1659.

[11] Louis Jeanjean, Stefan Le Coz, An existence and stability result for standing waves of nonlinear Schrödinger equations, Adv. Differ. Equ. 11 (7) (2006) 813–840.

[12] Louis Jeanjean, Sheng-Sen Lu, On global minimizers for a mass constrained problem, Calc. Var. Partial Differ. Equ. 61 (6) (2022) 214, 18 pp.

[13] Xiaosong Kang, Juncheng Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations, Adv. Differ. Equ. 5 (7–9) (2000) 899–928.

[14] Man Kam Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$, Arch. Ration. Mech. Anal. 105 (3) (1989) 243–266.

[15] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (4) (1984) 223–283.

[16] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (2) (1984) 109–145.

[17] Nicola Soave, Normalized ground states for the NLS equation with combined nonlinearities, J. Differ. Equ. 269 (9) (2020) 6941–6987.

[18] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 87 (4) (1982/83) 567–576.