

Article

On a Nonlinear Fractional Delay Differential Equation with Nonlocal Boundary Conditions

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Abstract

The aim of this paper is to investigate the existence, uniqueness, and Hyers–Ulam stability of solutions to a class of nonlinear Caputo fractional delay differential equation with nonlocal boundary conditions in the space of absolutely continuous functions. Owing to the nonlinearity, the equation is not readily amenable to treatment using existing integral transform methods. Our approach employs the inverse operator technique, the multivariate Mittag-Leffler function, and several well-established fixed-point theorems. An illustrative example is provided to demonstrate the applicability of the obtained conditions. A heuristic discussion of an exact solution, together with sufficient conditions justifying the convergence of the associated formal series in the Laplace-transform domain, is also presented.

Keywords: fractional delay differential equation; uniqueness; existence; Hyers–Ulam stability; fixed point theorem; multivariate Mittag-Leffler function; inverse operator

MSC: 26A33; 33E12; 34B10; 34K20

1. Introduction

Let $T > 0$, and let $C([0, T])$ be the Banach space of all real-valued continuous functions on $[0, T]$ with the norm

$$\|u\|_{\infty} := \max_{t \in [0, T]} |u(t)| < +\infty.$$

For $\alpha \in (0, 1)$ and $u \in C([0, T])$, the Riemann–Liouville fractional integral operator I_{0+}^{α} is defined by ([1]) p. 69, Equation (2.1.1))

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [0, T].$$

It follows from [2,3] that $I_{0+}^0 u = u$. In addition,

$$\|I_{0+}^{\alpha} u\|_{\infty} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|u\|_{\infty}, \quad (1)$$

provided that $u \in C([0, T])$.



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conditions of the [Creative Commons](https://creativecommons.org/licenses/by/4.0/)[Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

Let $C^1([0, T])$ denote the space of all real-valued continuously differentiable functions on $[0, T]$, and let $AC([0, T])$ denote the Banach space of all real-valued absolutely continuous functions on $[0, T]$ endowed with the standard norm

$$\|u\|_{AC} := \|u\|_{\infty} + \int_0^T |u'(t)| dt = \|u\|_{\infty} + \|u'\|_{L^1}.$$

We have $C^1([0, T]) \subsetneq AC([0, T]) \subsetneq C([0, T])$. However, the space $AC([0, T])$ equipped with $\|\cdot\|_{\infty}$ is not complete. It is particularly important to note that Riemann–Liouville fractional integral operator I_{0+}^{α} do not necessarily map $C([0, T])$ into $AC([0, T])$ ([4] p. 5, Proposition 2.9).

For $u \in AC([0, T])$, the Caputo fractional derivative operator ${}^C D_{0+}^{\alpha}$ of order $\alpha \in [0, 1]$ can be defined by ([1] p. 92, Theorem 2.1)

$${}^C D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds, \quad t \in [0, T].$$

Note that if $u \in AC([0, T])$, we have ${}^C D_{0+}^0 u(t) = u(t)$ and ${}^C D_{0+}^1 u(t) = u'(t)$. In addition, if $\alpha \in (0, 1]$ and $u \in AC([0, T])$ then ([1] p. 96, Lemma 2.22)

$$I_{0+}^{\alpha} ({}^C D_{0+}^{\alpha} u)(t) = u(t) - u(0).$$

Assume that $0 < \alpha_m < \alpha_{m-1} < \dots < \alpha_1 < 1$ for some $m \in \mathbb{N}$. In this paper, we focus on the following fractional multi-pantograph differential equation with nonlocal boundary conditions:

$$\begin{cases} u'(t) + a_1 {}^C D_{0+}^{\alpha_1} u(t) + \dots + a_m {}^C D_{0+}^{\alpha_m} u(t) = f(t, u(\lambda_1 t), \dots, u(\lambda_n t)), & t \in [0, T], \\ u(0) + \beta u(T) = 0, & \beta \text{ is a nonzero constant,} \end{cases} \quad (2)$$

(3)

where $a_1, \dots, a_m \in \mathbb{R}$, $0 < \lambda_1, \dots, \lambda_n \leq 1$ and $f \in C([0, T] \times \mathbb{R}^n)$.

The pantograph-type equations, typified by (see, for example, [5,6])

$$u'(t) = au(\lambda t), \quad (4)$$

are an important class of functional differential equations that has been studied for nearly a century. It has a nice series solution

$$u(t) = \sum_{k=0}^{\infty} a^k \lambda^{\frac{k(k-1)}{2}} \frac{t^k}{k!}. \quad (5)$$

These equations not only possess rich mathematical structures, but also have many applications. When we let $a_1 = \dots = a_m = 0$ in (2), it reduces to the equation

$$u'(t) = f(t, u(\lambda_1 t), \dots, u(\lambda_n t)), \quad (6)$$

which contains (4) and many other important known pantograph equations as special cases (see [7–10]).

Fractional extensions of (6) obtained by replacing the usual derivative with a particular fractional derivative have also been widely studied. For example, using the Banach contraction mapping principle, Balachandran et al. [11] have established the existence and uniqueness of solution of the following nonlinear fractional pantograph equation

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) = f(t, u(t), u(\lambda t)), \\ u(0) = u_0, \end{cases} \quad (7)$$

(8)

where $0 < \alpha < 1$, $0 < \lambda < 1$ and $f \in C([0, T] \times \mathbb{R}^2)$. Jalilian [12] proved the existence of solution for the fractional pantograph equation

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = f(t, u(t), u(\lambda_1 t), u(\lambda_2 t)), \\ u(0) = u_0, \end{cases}$$

where $0 < \alpha < 1$, $0 < \lambda_1, \lambda_2 < 1$ and $f \in C([0, T] \times \mathbb{R}^3)$. Very recently, Duan and Duan [13] derived an explicit series solution of the fractional pantograph equation

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = -au(t) - bu(\lambda t) + f(t), \\ u(0) = u_0, \end{cases}$$

by introducing a new Mittag-Leffler type function.

Except the above typical equations and their related classical initial value problems, there are some latest progress in this direction. Hafeez et al. [14] investigated the following fractional delay differential equation

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = f(t, u(t), u(g(t))), & t \in [0, T], \quad 1 < \alpha < 2, \\ u(t) = \omega(t) = \omega_0, \quad u'(t) = \omega'(t) = \omega_1, & t \in [-\gamma, 0], \end{cases}$$

where $\omega \in C([-\gamma, 0])$, $f \in C([0, T] \times \mathbb{R}^2)$, and $t \geq g(t)$. In addition, Hashim et al. [15] studied the following problem

$$\begin{cases} u''(t) = D_{0+}^{\alpha(t)} u(t) + a(t)u(\lambda t) + r(t), & t \in [0, 1], \\ u(0) = u_0, u'(0) = u_1, \end{cases}$$

where $0 < \lambda < 1$, $0 < \alpha(t) < 1$ for all $t \in [0, 1]$, $a, r \in C([0, 1])$, and $D_{0+}^{\alpha(t)}$ is the variable-order Caputo fractional derivative defined by

$$D_{0+}^{\alpha(t)} u(t) = \frac{1}{\Gamma(1 - \alpha(t))} \int_0^t (t-s)^{-\alpha(t)} u'(s) ds.$$

Fractional multi-pantograph differential equations have recently attracted considerable attention because they simultaneously incorporate hereditary effects through fractional derivatives and proportional delay through pantograph arguments. Such models arise naturally in the mathematical description of phenomena with memory and self-similar time delays, including population dynamics, control theory, viscoelasticity, and engineering systems. The presence of multiple fractional derivatives of different orders together with nonlocal boundary conditions significantly enriches the mathematical structure of the problem while also introducing substantial analytical challenges. Consequently, investigating the existence, uniqueness, and stability of solutions to problems (2) and (3) is of both theoretical interest and practical importance.

Definition 1. The boundary value problems (2) and (3) is said to be Hyers–Ulam stable if there exists a constant $A > 0$ such that, for every $\varepsilon > 0$ and every $u \in C^1([0, T])$ satisfying the nonlocal boundary condition

$$u(0) + \beta u(T) = 0$$

and

$$\left\| u'(t) + a_1 {}^C D_{0+}^{\alpha_1} u(t) + \cdots + a_m {}^C D_{0+}^{\alpha_m} u(t) - f(t, u(\lambda_1 t), \dots, u(\lambda_n t)) \right\|_\infty \leq \varepsilon,$$

there exists an exact solution $u_0 \in AC([0, T])$ of the boundary value problems (2) and (3) satisfying

$$\|u - u_0\|_\infty \leq A\varepsilon,$$

where A is independent of u_0, u and ε . We call such A a Hyers–Ulam stability constant.

The remainder of the paper is organized as follows. In Section 2, we first establish several necessary properties of the multivariate Mittag-Leffler function, and then apply the inverse operator method to derive an equivalent integral equation for nonlocal problems (2) and (3). In Section 3, we first use fixed-point theory to characterize the existence and uniqueness of the solutions in $AC([0, T])$ to the problems (2) and (3) under different conditions, and then present an analysis of its Hyers–Ulam stability. Finally, some concluding remarks and directions for future research are included in Section 4.

2. Preliminaries and Auxiliary Results

2.1. The Multivariate Mittag-Leffler Function

Let $\alpha_1, \dots, \alpha_m, \beta > 0$ and $z_1, \dots, z_m \in \mathbb{C}$. Hadid and Luchko [16] introduced the following multivariate Mittag-Leffler function

$$E_{(\alpha_1, \dots, \alpha_m), \beta}(z_1, \dots, z_m) := \sum_{k=0}^{\infty} \sum_{k_1 + \dots + k_m = k} \binom{k}{k_1, \dots, k_m} \frac{z_1^{k_1} \dots z_m^{k_m}}{\Gamma(\alpha_1 k_1 + \dots + \alpha_m k_m + \beta)}, \quad (9)$$

which is an entire function on $\mathbb{C}^m$. In recent years, this function has appeared frequently in many studies (see, for example, [17–19]), and its position within fractional calculus is becoming ever more significant.

When $m = 1$, the function (9) reduces to the following two-parameter Mittag-Leffler function:

$$E_{\alpha, \beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C}. \quad (10)$$

Further letting $\beta = 1$ in (10), we obtain the classical Mittag-Leffler function defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathbb{C}. \quad (11)$$

Throughout this paper, for convenience of notation, we follow Bazhlekova [20] and let

$$\mathcal{E}_{(\alpha_1, \dots, \alpha_m), \beta}(t; a_1, \dots, a_m) := t^{\beta-1} E_{(\alpha_1, \dots, \alpha_m), \beta}(-a_1 t^{\alpha_1}, \dots, -a_m t^{\alpha_m}).$$

From ([20] Theorem 2.3, Equation (2.11)), we have immediately

$$\left(\frac{d}{dt}\right)^n \mathcal{E}_{(\alpha_1, \dots, \alpha_m), \beta}(t; a_1, \dots, a_m) = \mathcal{E}_{(\alpha_1, \dots, \alpha_m), \beta-n}(t; a_1, \dots, a_m), \quad \beta > n. \quad (12)$$

In particular, we shall use the following relation:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_{(\alpha_1, \dots, \alpha_m), 1}(t; a_1, \dots, a_m) &= \frac{d}{dt} E_{(\alpha_1, \dots, \alpha_m), 1}(-a_1 t^{\alpha_1}, \dots, -a_m t^{\alpha_m}) \\ &= \sum_{k=1}^{\infty} \sum_{k_1 + \dots + k_m = k} \binom{k}{k_1, \dots, k_m} \frac{(-a_1)^{k_1} \dots (-a_m)^{k_m} t^{\alpha_1 k_1 + \dots + \alpha_m k_m - 1}}{\Gamma(\alpha_1 k_1 + \dots + \alpha_m k_m)} \\ &= \mathcal{O}(t^{\sigma-1}), \quad t \rightarrow 0^+, \end{aligned} \quad (13)$$

where $0 < \sigma := \min\{\alpha_1, \dots, \alpha_m\} < 1$. From the asymptotic behaviour (13), it follows immediately that

$$\mathcal{E}_{(\alpha_1, \dots, \alpha_m), 1}(t; a_1, \dots, a_m) \in AC([0, T]). \quad (14)$$

The following technical result is crucial for our derivation in Section 2.2, where we shall consider the convolution of $\mathcal{E}_{(\alpha_1, \dots, \alpha_m), 1}(t; a_1, \dots, a_m)$ with a continuous function. Although it may be known, we have not been able to find an explicit reference. For completeness and reader's convenience, we provide a full proof here.

Proposition 1. Let $k \in AC([0, T])$ and $f \in C([0, T])$. Define

$$g(t) := (k * f) = \int_0^t k(t-s)f(s)ds.$$

Then $g \in AC([0, T])$.

Proof. In view of $f \in C([0, T])$, there is a positive number M so that $|f(s)| \leq M$ for all $s \in [0, T]$. By the Leibniz rule for differentiation under the integral sign, we have

$$g'(t) = \int_0^t k'(t-s)f(s)ds + k(0)f(t).$$

Next, we show that $g' \in L^1(0, T)$. First note that

$$\begin{aligned} \int_0^T |g'(t)|dt &\leq \int_0^T \left[\int_0^t |k'(t-s)||f(s)|ds \right] dt + |k(0)| \int_0^T |f(t)|dt \\ &\leq M \int_0^T \left[\int_0^t |k'(t-s)|ds \right] dt + |k(0)|MT. \end{aligned} \quad (15)$$

Since $k \in AC([0, T])$, we have $k' \in L^1(0, T)$, and thus

$$\begin{aligned} \int_0^T \left[\int_0^t |k'(t-s)|ds \right] dt &= \int_0^T \left[\int_0^t |k'(\tau)|d\tau \right] dt \\ &= \int_0^T (T-\tau)|k'(\tau)|d\tau \leq T\|k'\|_{L^1} < \infty. \end{aligned} \quad (16)$$

Combining (15) and (16) gives

$$\|g'\|_{L^1} = \int_0^T |g'(t)|dt < \infty,$$

which completes the proof of our assertion. $\square$

From the complete monotonicity of $E_{\alpha, \beta}(-t)$ with $t \geq 0$ (which means that $(-1)^n \frac{d^n}{dt^n} E_{\alpha, \beta}(-t) \geq 0$ for $n = 0, 1, 2, \dots$), we have

$$E_{\alpha, \beta}(-t) \leq E_{\alpha, \beta}(0) = \frac{1}{\Gamma(\beta)}, \quad (17)$$

where $t \geq 0$, $0 < \alpha \leq 1$ and $\beta \geq \alpha$. In some studies (e.g., [21]), inequality (17) allows the obtained conditions to be more concise. Bazhlekova also showed that if $0 < \alpha_m < \dots < \alpha_1 \leq \beta \leq 1$ and $a_j > 0$ ($j = 1, \dots, m$),

$$\mathcal{E}_{(\alpha_1, \dots, \alpha_m), \beta}(t; a_1, \dots, a_m)$$

is completely monotonic for $t > 0$ (see ([20] p. 95, Theorem 3.2)). Thus, it is non-negative and non-increasing on $(0, +\infty)$. Considering that $\mathcal{E}_{(\alpha_1, \dots, \alpha_m), \beta}(t; a_1, \dots, a_m)$ behaves as $t^{\beta-1}/\Gamma(\beta)$ as $t \rightarrow 0$, we cannot obtain an inequality of the form (17) as in the case of the two-parameter Mittag-Leffler function, unless when $\beta = 1$, we have

$$\mathcal{E}_{(\alpha_1, \dots, \alpha_m), 1}(t; a_1, \dots, a_m) \leq 1.$$

In addition, Luo et al. [18] recently obtained the following inequality which we state here as a proposition.

Proposition 2. Let $\alpha_j > 0$ ($j = 1, \dots, m$), and let x_0 be a positive number so that

$$\Gamma(x_0) = \min_{x \in (0, +\infty)} \Gamma(x).$$

Then, for $\beta \geq x_0$,

$$|E_{(\alpha_1, \dots, \alpha_m), \beta}(z_1, \dots, z_m)| \leq E_{\min \alpha_j, \beta}(|z_1| + \dots + |z_m|),$$

where $(z_1, \dots, z_m) \in \mathbb{C}^m$.

Numerical computation yields $x_0 = 1.46163 \dots$. If $z_1, \dots, z_m > 0$, we have

$$E_{(\alpha_1, \dots, \alpha_m), \beta}(z_1, \dots, z_m) \leq E_{\min \alpha_j, \beta}(z_1 + \dots + z_m), \quad (18)$$

where $\alpha_j > 0$ ($j = 1, \dots, m$) and $\beta \geq x_0$. This inequality reduces the multivariate Mittag-Leffler function to a two-parameter one, allowing for direct numerical computation with existing codes online.

2.2. Construction of an Equivalent Integral Equation: An Inverse Operator Method

The inverse operator method offers a robust and efficient framework for solving various fractional differential and integral equations, as well as for analyzing existence, uniqueness, and stability via fixed-point theory (see, for example [18,22–25]).

In this subsection, we reformulate the boundary value problems (2) and (3) as an equivalent implicit integral equation in $AC([0, T])$ under the condition:

$$Y \equiv \frac{1}{\beta} + 1 + \sum_{i=1}^m \frac{a_i T^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \neq 0. \quad (19)$$

This formulation will serve as the foundation for the subsequent analysis of existence, uniqueness, and stability.

Recall that

$$\begin{aligned} I_{0+}^1 ({}^C D_{0+}^\alpha u(t)) &= I_{0+}^{1-\alpha} (I_{0+}^\alpha ({}^C D_{0+}^\alpha u(t))) \\ &= I_{0+}^{1-\alpha} (u(t) - u(0)) = I_{0+}^{1-\alpha} u(t) - \frac{u(0)}{\Gamma(2-\alpha)} t^{1-\alpha}. \end{aligned}$$

Applying the operator I_{0+}^1 to both sides of Equation (2) yields

$$u(t) - u(0) + \sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} u(t) - u(0) \sum_{i=1}^m \frac{a_i}{\Gamma(2-\alpha_i)} t^{1-\alpha_i} = I_{0+}^1 f(t, u(\lambda_1 t), \dots, u(\lambda_n t)), \quad (20)$$

by noting that $u \in AC([0, T])$. Setting $t = T$ in (20), we have

$$u(T) - u(0) \left[1 + \sum_{i=1}^m \frac{a_i}{\Gamma(2 - \alpha_i)} T^{1-\alpha_i} \right] + \sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} u(T) = \int_0^T f(s, u(\lambda_1 s), \dots, u(\lambda_n s)) ds.$$

Using the boundary condition $u(0) + \beta u(T) = 0$ with $\beta \neq 0$ and substituting $u(T)$, we finally have

$$u(0) = \frac{1}{Y} \sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} u(T) - \frac{1}{Y} \int_0^T f(s, u(\lambda_1 s), \dots, u(\lambda_n s)) ds. \quad (21)$$

Clearly, $u(0)$ is nonlocal.

Hence, from Equation (20) we get

$$\left(1 + \sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} \right) u(t) = I_{0+}^1 f(t, u(\lambda_1 t), \dots, u(\lambda_n t)) + u(0) \left[1 + \sum_{i=1}^m \frac{a_i}{\Gamma(2 - \alpha_i)} t^{1-\alpha_i} \right]. \quad (22)$$

Define an operator

$$\begin{aligned} P &= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} \right)^k \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1 + \dots + k_m = k} \binom{k}{k_1, \dots, k_m} a_1^{k_1} \dots a_m^{k_m} I_{0+}^{k_1(1-\alpha_1) + \dots + k_m(1-\alpha_m)}, \end{aligned}$$

by using the semigroup property of the Riemann–Liouville fractional integral operator I_{0+}^α . We shall prove that P is a well-defined operator from $C([0, T])$ to itself. Indeed, for any $\varphi \in C([0, T])$ we have from (1) that

$$\begin{aligned} \|P\varphi\|_{\infty} &\leq \|\varphi\|_{\infty} \sum_{k=0}^{\infty} \sum_{k_1 + \dots + k_m = k} \binom{k}{k_1, \dots, k_m} \frac{|a_1|^{k_1} \dots |a_m|^{k_m} T^{k_1(1-\alpha_1) + \dots + k_m(1-\alpha_m)}}{\Gamma(1 + k_1(1-\alpha_1) + \dots + k_m(1-\alpha_m))} \\ &= \|\varphi\|_{\infty} E_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(|a_1| T^{1-\alpha_1}, \dots, |a_m| T^{1-\alpha_m}) < +\infty. \end{aligned}$$

Furthermore, P is a unique inverse operator of $1 + a_1 I_{0+}^{1-\alpha_1} + \dots + a_m I_{0+}^{1-\alpha_m}$. To see this, we note that

$$\begin{aligned} P \left(1 + a_1 I_{0+}^{1-\alpha_1} + \dots + a_m I_{0+}^{1-\alpha_m} \right) &= P + \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} \right)^{k+1} \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k \left(\sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} \right)^k + \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^m a_i I_{0+}^{1-\alpha_i} \right)^{k+1} = 1 \quad (\text{identity operator}). \end{aligned}$$

Similarly,

$$\left(1 + a_1 I_{0+}^{1-\alpha_1} + \dots + a_m I_{0+}^{1-\alpha_m} \right) P = 1,$$

and the uniqueness of P follows easily.

Applying the operator P to Equation (22), we deduce that

$$u(t) = P I_{0+}^1 f(t, u(\lambda_1 t), \dots, u(\lambda_n t)) + u(0) \left[P 1 + \sum_{i=1}^m \frac{a_i}{\Gamma(2 - \alpha_i)} P t^{1-\alpha_i} \right]. \quad (23)$$

Note that for $\rho > -1$,

$$\begin{aligned}
Pt^\rho &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\dots+k_m=k} \binom{k}{k_1, \dots, k_m} a_1^{k_1} \dots a_m^{k_m} I_{0+}^{k_1(1-\alpha_1)+\dots+k_m(1-\alpha_m)} t^\rho \\
&= \Gamma(\rho+1) t^\rho \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\dots+k_m=k} \binom{k}{k_1, \dots, k_m} \frac{a_1^{k_1} \dots a_m^{k_m} t^{k_1(1-\alpha_1)+\dots+k_m(1-\alpha_m)}}{\Gamma(\rho+1+k_1(1-\alpha_1)+\dots+k_m(1-\alpha_m))} \\
&= \Gamma(\rho+1) t^\rho E_{(1-\alpha_1, \dots, 1-\alpha_m), \rho+1}(-a_1 t^{1-\alpha_1}, \dots, -a_m t^{1-\alpha_m}) \\
&= \Gamma(\rho+1) \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), \rho+1}(t; a_1, \dots, a_m).
\end{aligned}$$

So we have

$$\begin{aligned}
u(0) \left[P1 + \sum_{i=1}^m \frac{a_i}{\Gamma(2-\alpha_i)} Pt^{1-\alpha_i} \right] &= u(0) \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t; a_1, \dots, a_m) \\
&\quad + u(0) \sum_{i=1}^m a_i \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 2-\alpha_i}(t; a_1, \dots, a_m). \quad (24)
\end{aligned}$$

In addition, we have

$$\begin{aligned}
&PI_{0+}^1 f(t, u(\lambda_1 t), \dots, u(\lambda_n t)) \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\dots+k_m=k} \binom{k}{k_1, \dots, k_m} a_1^{k_1} \dots a_m^{k_m} I_{0+}^{k_1(1-\alpha_1)+\dots+k_m(1-\alpha_m)+1} f(t, u(\lambda_1 t), \dots, u(\lambda_n t)) \\
&= \int_0^t E_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(-a_1(t-s)^{1-\alpha_1}, \dots, -a_m(t-s)^{1-\alpha_m}) f(s, u(\lambda_1 s), \dots, u(\lambda_n s)) ds \\
&= \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) f(s, u(\lambda_1 s), \dots, u(\lambda_n s)) ds. \quad (25)
\end{aligned}$$

Substituting (24) and (25) into (23) gives

$$\begin{aligned}
u(t) &= \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) f(s, u(\lambda_1 s), \dots, u(\lambda_n s)) ds \\
&\quad + u(0) \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t; a_1, \dots, a_m) \\
&\quad + u(0) \sum_{i=1}^m a_i \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 2-\alpha_i}(t; a_1, \dots, a_m). \quad (26)
\end{aligned}$$

For convenience, we let

$$a_0 \equiv 1, \quad \alpha_0 \equiv 1.$$

Then the above equation can be written as the following more compact form:

$$\begin{aligned}
u(t) &= \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) f(s, u(\lambda_1 s), \dots, u(\lambda_n s)) ds \\
&\quad + u(0) \sum_{i=0}^m a_i \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 2-\alpha_i}(t; a_1, \dots, a_m) \\
&= \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) F_u(s) ds \\
&\quad + \frac{H(t)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} u(T) - \int_0^T F_u(s) ds \right]. \quad (27)
\end{aligned}$$

where

$$F_u(s) := f(s, u(\lambda_1 s), \dots, u(\lambda_n s)),$$

and

$$H(t) := \sum_{i=0}^m a_i \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 2-\alpha_i}(t; a_1, \dots, a_m).$$

Consequently, problems (2) and (3) is equivalent to the integral Equation (27). Indeed, Equation (27) is obtained from (20) by applying the inverse operator P and substituting the expression (21) for $u(0)$, which is determined by the nonlocal boundary condition (3). Conversely, reversing the preceding calculations shows that every solution of (27) satisfies Equation (2) together with the nonlocal boundary condition (3). Thus, unlike an integral representation in which $u(0)$ is treated as an independent quantity, Equation (27) explicitly incorporates the boundary condition.

Clearly, Equation (27) is an implicit integral equation. It remains to be shown that the right-hand side of (27) belongs to $AC([0, T])$ provided that $u \in C([0, T])$. Since

$$\frac{d}{dt} \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 2-\alpha_i}(t; a_1, \dots, a_m) = \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1-\alpha_i}(t; a_1, \dots, a_m) \in L^1(0, T),$$

we have immediately

$$\frac{H(t)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} u(T) - \int_0^T F_u(s) ds \right] \in AC([0, T]).$$

The fact that

$$\int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) F_u(s) ds \in AC([0, T])$$

follows from the fact (14) and Proposition 1.

3. Main Results

3.1. Existence and Uniqueness by the Banach Contraction Principle

We are now ready to present the following theorem regarding the existence and uniqueness of solutions to the boundary value problems (2) and (3) in $AC([0, T])$.

Bear in mind that, in what follows, Y is defined by (19) and

$$a_0 \equiv 1, \quad \alpha_0 \equiv 1.$$

Theorem 1. Assume that $f \in C([0, T] \times \mathbb{R}^n)$ satisfies the Lipschitz condition for $L \geq 0$:

$$|f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)| \leq L \sum_{j=1}^n |x_j - y_j|, \quad t \in [0, T], \quad x_j, y_j \in \mathbb{R}, \quad (28)$$

and $Y \neq 0$. Furthermore, we suppose that

$$K := LnTL_0 + \frac{H_0}{|Y|} (Q + LnT) < 1, \quad (29)$$

where

$$L_i := \max_{t \in [0, T]} |\mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 2-\alpha_i}(t; a_1, \dots, a_m)|, \quad i = 0, 1, \dots, m,$$

and

$$H_0 := \sum_{i=0}^m |a_i| L_i, \quad Q := \sum_{i=1}^m \frac{|a_i| T^{1-\alpha_i}}{\Gamma(2-\alpha_i)}.$$

Then the boundary value problems (2) and (3) has a unique solution in $AC([0, T])$.

Proof. We begin defining a nonlinear mapping Φ over $C([0, T])$ as

$$\begin{aligned}\Phi u(t) := & \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) F_u(s) ds \\ & + \frac{H(t)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} u(T) - \int_0^T F_u(s) ds \right].\end{aligned}\quad (30)$$

It follows from Section 2.1 that $\Phi : C([0, T]) \rightarrow AC([0, T])$. For any $u, v \in C([0, T])$, we have

$$\left| \sum_{i=1}^m a_i I_0^{1-\alpha_i} (u-v)(T) \right| \leq Q \|u-v\|_\infty,$$

and

$$\left| \int_0^T F_u(s) ds - \int_0^T F_v(s) ds \right| \leq LnT \|u-v\|_\infty,$$

using the Lipschitz condition (28). Furthermore,

$$\begin{aligned}\Phi u(t) - \Phi v(t) = & \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) [F_u(s) - F_v(s)] ds \\ & + \frac{H(t)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} (u-v)(T) - \int_0^T (F_u(s) - F_v(s)) ds \right].\end{aligned}$$

Therefore,

$$\|\Phi u - \Phi v\|_\infty \leq LnTL_0 \|u-v\|_\infty + \frac{H_0}{|Y|} (Q + LnT) \|u-v\|_\infty = K \|u-v\|_\infty, \quad (31)$$

where K is given by (29). Since $K < 1$, the boundary value problems (2) and (3) has a unique solution in $C([0, T])$ by the Banach contraction principle. Moreover, the mapping Φ is from $C([0, T])$ to $AC([0, T])$. Therefore, this unique solution in fact belongs to $AC([0, T])$. Our approach here is indeed much simpler than working directly with the norm $\|u\|_{AC}$, which contains the second term $\|u'\|_{L^1}$. $\square$

Corollary 1. Let $0 < \alpha < 1$, $0 < \lambda \leq 1$, and consider the single-order fractional pantograph boundary value problem

$$\begin{cases} u'(t) + a {}^C D_{0+}^\alpha u(t) = f(t, u(\lambda t)), & t \in [0, T], \\ u(0) + \beta u(T) = 0, \end{cases}$$

where $a \in \mathbb{R}$, $\beta \neq 0$, and $f \in C([0, T] \times \mathbb{R})$. Suppose that

$$|f(t, x) - f(t, y)| \leq L_f |x - y|, \quad \text{and} \quad Y = \frac{1}{\beta} + 1 + \frac{aT^{1-\alpha}}{\Gamma(2-\alpha)} \neq 0.$$

Define

$$L_0 = \max_{t \in [0, T]} |E_{1-\alpha, 1}(-at^{1-\alpha})|, \quad L_1 = \max_{t \in [0, T]} |t^{1-\alpha} E_{1-\alpha, 2-\alpha}(-at^{1-\alpha})|,$$

and

$$H_0 = L_0 + |a|L_1, \quad Q = \frac{|a|T^{1-\alpha}}{\Gamma(2-\alpha)}.$$

If

$$K = L_f TL_0 + \frac{H_0}{|Y|} (Q + L_f T) < 1,$$

then the above boundary value problem has a unique solution in $AC([0, T])$.

Proof. The assertion follows immediately from Theorem 1 by taking $m = n = 1$, $a_1 = a$, $\alpha_1 = \alpha$, and $\lambda_1 = \lambda$. $\square$

This corollary shows explicitly how the general multi-order and multi-delay result reduces to the single-order and single-delay setting. It is related to the classical fractional pantograph problems discussed in Section 1, including that of Balachandran et al. [11]. The present formulation, however, contains both the first-order derivative and a Caputo fractional derivative and is subject to a nonlocal boundary condition; hence it is not identical to the initial value problem studied in [11].

3.2. Existence by Leray–Schauder Fixed Point Theorem

In the previous section, we have established an existence and uniqueness result based on Banach's fixed point theorem by requiring that f satisfies a Lipschitz condition. Here, we will employ the Leray–Schauder fixed point theorem to provide an existence result from another perspective.

Theorem 2 (Leray–Schauder fixed point theorem). *Let T be a continuous and compact mapping of a Banach space X to itself, such that the set*

$$\{x \in X : x = \delta Tx \text{ for some } \delta \in [0, 1]\}$$

is bounded. Then T has a fixed point.

Theorem 3. *Assume that $f \in C([0, T] \times \mathbb{R}^n)$ satisfies the following linear growth condition for nonnegative numbers c_0 and c_1 :*

$$|F_u(t)| = |f(t, u(\lambda_1 t), \dots, u(\lambda_n t))| \leq c_0 + c_1 \|u\|_\infty, \quad u \in C([0, T]), \quad (32)$$

and $Y \neq 0$. Also, assume that

$$K_0 = TL_0 c_1 + \frac{H_0}{|Y|} (Q + Tc_1) < 1. \quad (33)$$

Then the boundary value problems (2) and (3) has a solution in $AC([0, T])$.

Proof. Again we use the nonlinear mapping Φ over $C([0, T])$ defined by (30). The proof is divided into four steps.

Step 1. The mapping $\Phi : C([0, T]) \rightarrow C([0, T])$ is continuous. This follows from the following inequalities (note that f is not a Lipschitz function here):

$$\begin{aligned} & \|\Phi u - \Phi v\|_\infty \\ & \leq T \max_{t \in [0, T]} |F_u(t) - F_v(t)| \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(T; -|a_1|, \dots, -|a_m|) \\ & \quad + \frac{H_0}{|Y|} \left[Q \|u - v\|_\infty + T \max_{t \in [0, T]} |F_u(t) - F_v(t)| \right], \end{aligned}$$

which imply Φ is continuous since $f \in C([0, T] \times \mathbb{R}^n)$.

Step 2. We show that Φ is a mapping from a bounded set to a bounded set in $C([0, T])$. Let W_0 be a bounded set in $C([0, T])$, namely,

$$W_0 := \{u \in C([0, T]); \|u\|_\infty \leq M\},$$

where M is a positive number. For any $u \in W_0$, the function $t \mapsto f(t, u(\lambda_1 t), \dots, u(\lambda_n t))$ is continuous and bounded on $[0, T]$. Indeed,

$$M_f := \max_{t \in [0, T]} |f(t, u(\lambda_1 t), \dots, u(\lambda_n t))| \leq c_0 + c_1 \|u\|_\infty < \infty.$$

Then for any $u \in W_0$, we have

$$\|\Phi u\|_\infty \leq TL_0 M_f + \frac{H_0}{|Y|} [QM + TM_f].$$

Therefore, Φu is uniformly bounded for all $u \in W_0$.

Step 3. We show that Φ is completely continuous as a mapping from $C([0, T])$ into itself. By the Arzelà–Ascoli theorem, it remains to prove that Φ maps every bounded subset $W_0 \subset C([0, T])$ into a family of functions that is equicontinuous (and uniformly bounded, which was done in *Step 2*). By definition of equicontinuity, we need to show that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall t_1, t_2 \in [0, T] \forall u \in W_0 : |t_1 - t_2| < \delta \Rightarrow |(\Phi u)(t_2) - (\Phi u)(t_1)| < \varepsilon.$$

Observe that

$$\begin{aligned} \Phi u(t_2) - \Phi u(t_1) &:= \int_0^{t_2} \widehat{\mathcal{E}}_0(t_2 - s) F_u(s) ds - \int_0^{t_1} \widehat{\mathcal{E}}_0(t_1 - s) F_u(s) ds \\ &\quad + \frac{H(t_2) - H(t_1)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} u(T) - \int_0^T F_u(s) ds \right], \end{aligned} \quad (34)$$

where we denote

$$\widehat{\mathcal{E}}_i(t) \equiv \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 2-\alpha_i}(t; a_1, \dots, a_m) \quad (i = 0, 1, \dots, m)$$

for convenience. Since $\widehat{\mathcal{E}}_i(t) \in AC([0, T])$, $\widehat{\mathcal{E}}_i(t)$ is uniformly continuous, and thus

$$\forall \varepsilon > 0 \exists \delta > 0 \forall t_1, t_2 \in [0, T] \forall u \in W_0 : |t_1 - t_2| < \delta \Rightarrow |\widehat{\mathcal{E}}_i(t_2) - \widehat{\mathcal{E}}_i(t_1)| < \varepsilon,$$

(note that all $\widehat{\mathcal{E}}_i(t)$ are independent of u). Clearly,

$$\begin{aligned} &\left| \frac{H(t_2) - H(t_1)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} u(T) - \int_0^T F_u(s) ds \right] \right| \\ &\leq \frac{1}{|Y|} [QM + TM_f] \sum_{i=0}^m |a_i| \varepsilon. \end{aligned}$$

Without loss of generality, let $t_1, t_2 \in [0, T]$ satisfy $t_2 > t_1$ and $|t_1 - t_2| < \delta$. We have

$$\begin{aligned} &\left| \int_0^{t_2} \widehat{\mathcal{E}}_0(t_2 - s) F_u(s) ds - \int_0^{t_1} \widehat{\mathcal{E}}_0(t_1 - s) F_u(s) ds \right| \\ &= \int_0^{t_1} |\widehat{\mathcal{E}}_0(t_2 - s) - \widehat{\mathcal{E}}_0(t_1 - s)| |F_u(s)| ds + \int_{t_1}^{t_2} |\widehat{\mathcal{E}}_0(t_2 - s)| |F_u(s)| ds \\ &< \varepsilon \cdot TM_f + |t_2 - t_1| \cdot M_f \max_{s \in [0, T]} |\widehat{\mathcal{E}}_0(s)|. \end{aligned}$$

In summary, Φ is completely continuous.

Step 4. Finally, we show that for some $\delta \in [0, 1]$, the set

$$W = \{u \in C([0, T]) : u = \delta \Phi u\}$$

is bounded. In fact,

$$\|u\|_{\infty} \leq \|\Phi u\|_{\infty} \leq TL_0(c_0 + c_1\|u\|_{\infty}) + \frac{H_0}{|Y|}[Q\|u\|_{\infty} + T(c_0 + c_1\|u\|_{\infty})],$$

which implies that

$$(1 - K_0)\|u\|_{\infty} \leq Tc_0(L_0 + H_0/|Y|).$$

Thus,

$$\|u\|_{\infty} \leq \frac{Tc_0(L_0 + H_0/|Y|)}{1 - K_0}$$

is uniformly bounded. Hence, by the Leray–Schauder fixed-point theorem, the boundary value problems (2) and (3) has a solution in $C([0, T])$. By the same argument as in the proof of Theorem 1, this solution is absolutely continuous. This completes the proof. $\square$

Remark 1. We assume f satisfies a Lipschitz condition in Theorem 1, but only a linear growth in Theorem 3. Clearly, if f satisfies a Lipschitz condition, then it also satisfies a linear growth condition (see Section 3.3 in detail). However, the converse is generally false. For example, the function $f(t, y) = \sqrt{|y|} \leq 1 + |y|$ satisfies the linear growth condition, but is not a Lipschitz function.

3.3. Hyers–Ulam Stability

Theorem 4. Assume that $f \in C([0, T] \times \mathbb{R}^n)$ satisfies the Lipschitz condition (28), $Y \neq 0$, and

$$K := LnTL_0 + \frac{H_0}{|Y|}(Q + LnT) < 1.$$

Then the boundary value problems (2) and (3) is Hyers–Ulam stable. In particular, a Hyers–Ulam stability constant is

$$A = \frac{T(L_0 + H_0/|Y|)}{1 - K}.$$

Proof. For each $\varepsilon > 0$, we suppose that $u \in C^1([0, T])$ satisfies

$$\left\| u'(t) + a_1 {}^C D_{0+}^{\alpha_1} u(t) + \cdots + a_m {}^C D_{0+}^{\alpha_m} u(t) - F_u(t) \right\|_{\infty} \leq \varepsilon.$$

We let

$$\theta_u(t) = u'(t) + a_1 {}^C D_{0+}^{\alpha_1} u(t) + \cdots + a_m {}^C D_{0+}^{\alpha_m} u(t) - F_u(t),$$

so that $\|\theta_u\|_{\infty} \leq \varepsilon$. Clearly,

$$u'(t) + a_1 {}^C D_{0+}^{\alpha_1} u(t) + \cdots + a_m {}^C D_{0+}^{\alpha_m} u(t) = \theta_u(t) + F_u(t). \quad (35)$$

Applying the same argument used to derive the equivalent integral Equation (27) to Equation (35), and taking into account the nonlocal boundary condition $u(0) + \beta u(T) = 0$, we obtain

$$\begin{aligned} u(t) = & \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m)(F_u(s) + \theta_u(s))ds \\ & + \frac{H(t)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} u(T) - \int_0^T (F_u(s) + \theta_u(s))ds \right], \end{aligned}$$

By Theorem 1, there exists a unique solution $u_0 \in AC([0, T])$ of the boundary value problems (2) and (3), which satisfies the equivalent integral equation

$$u_0(t) = \int_0^t \mathcal{E}_{(1-\alpha_1, \dots, 1-\alpha_m), 1}(t-s; a_1, \dots, a_m) F_{u_0}(s) ds + \frac{H(t)}{Y} \left[\sum_{i=1}^m a_i I_0^{1-\alpha_i} u_0(T) - \int_0^T F_{u_0}(s) ds \right].$$

Thus, we have

$$|u(t) - u_0(t)| \leq TL_0 \varepsilon + LnTL_0 \|u - u_0\|_\infty + \frac{H_0}{|Y|} [Q \|u - u_0\|_\infty + LnT \|u - u_0\|_\infty + T\varepsilon].$$

Therefore,

$$\|u - u_0\|_\infty \leq A\varepsilon,$$

where

$$A := \frac{T(L_0 + H_0/|Y|)}{1 - K},$$

which is independent of u, u_0 and ε . This completes the proof. $\square$

3.4. An Illustrative Example

Example 1. Consider the following fractional pantograph boundary value problem:

$$\begin{cases} u'(t) + {}^C D_{0+}^{1/2} u(t) = 0.1u(t/2) + t, & t \in [0, 0.01], \\ u(0) + u(0.01) = 0. \end{cases}$$

In this case,

$$m = n = 1, \quad \alpha_1 = \frac{1}{2}, \quad a_1 = 1, \quad \lambda_1 = \frac{1}{2}, \quad \beta = 1, \quad T = 0.01,$$

and

$$f(t, x) = 0.1x + t.$$

Clearly,

$$|f(t, x) - f(t, y)| = 0.1|x - y|,$$

so the Lipschitz condition (28) holds with $L = 0.1$. Furthermore,

$$Y = 2 + \frac{(0.01)^{1/2}}{\Gamma(3/2)} \approx 2.112838 \neq 0.$$

Since $E_{1/2, 1}(-t^{1/2})$ is nonnegative and non-increasing for $t \geq 0$, we have

$$L_0 = \max_{t \in [0, 0.01]} |E_{1/2, 1}(-t^{1/2})| = 1.$$

Moreover,

$$L_1 = \max_{t \in [0, 0.01]} |t^{1/2} E_{1/2, 3/2}(-t^{1/2})| \approx 0.103543.$$

Therefore,

$$H_0 = L_0 + L_1 \approx 1.103543$$

and

$$Q = \frac{(0.01)^{1/2}}{\Gamma(3/2)} \approx 0.112838.$$

Consequently,

$$\begin{aligned} K &= 0.1(0.01)L_0 + \frac{H_0}{|Y|}(Q + 0.1(0.01)) \\ &\approx 0.060458 < 1. \end{aligned}$$

Thus, all the assumptions of Theorem 1 are satisfied, and the above boundary value problem has a unique solution in $AC([0, 0.01])$. In addition,

$$|f(t, u(t/2))| \leq 0.01 + 0.1\|u\|_\infty, \quad t \in [0, 0.01].$$

Hence, the linear growth condition in Theorem 3 holds with $c_0 = 0.01$ and $c_1 = 0.1$. In this case,

$$\begin{aligned} K_0 &= TL_0c_1 + \frac{H_0}{|Y|}(Q + Tc_1) \\ &\approx 0.060458 < 1. \end{aligned}$$

Therefore, Theorem 3 also guarantees the existence of a solution. Finally, by Theorem 4, the boundary value problem is Hyers–Ulam stable. In particular, a Hyers–Ulam stability constant is

$$\begin{aligned} A &= \frac{T(L_0 + H_0/|Y|)}{1 - K} \\ &\approx 0.01620. \end{aligned}$$

Thus, this example illustrates that the sufficient conditions in Theorems 1, 3 and 4 can be explicitly verified.

4. Concluding Remarks

Employing the inverse operator method, the multivariate Mittag–Leffler function, and several well-known fixed-point theorems, we have investigated the existence, uniqueness, and Hyers–Ulam stability of solutions to the fractional multi-pantograph differential Equation (2) with nonlocal boundary conditions in $AC([0, T])$. An illustrative example has also been provided to demonstrate that the sufficient conditions obtained in the main theorems can be explicitly verified.

Now a natural question arises: *Is it possible to find an explicit solution to the Equation (2) with (3)?* A complete rigorous treatment of the above problem lies beyond the scope of the present work. Here we provide a formal derivation for a particular case and subsequently establish sufficient conditions that justify the convergence of the resulting transform-domain series.

Consider the problem for $0 < \lambda < 1$:

$$\begin{cases} u'(t) - a {}^C D_{0+}^\alpha u(t) = bu(\lambda t), & t \in [0, T], \\ u(0) + \beta u(T) = 0, & \beta \text{ is a nonzero constant.} \end{cases} \quad (36)$$

$$(37)$$

An integral representation, which can be easily obtained by specializing (26), is given by

$$\begin{aligned} u(t) &= b \int_0^t E_{1-\alpha,1}(a(t-s)^{1-\alpha})u(\lambda s)ds \\ &\quad + u(0)E_{1-\alpha,1}(at^{1-\alpha}) + u(0)a \cdot t^{1-\alpha}E_{1-\alpha,2-\alpha}(at^{1-\alpha}), \end{aligned} \quad (38)$$

where $u(0)$ is determined by the boundary condition.

Applying the Laplace transform $\mathcal{L}$ to Equation (38) and taking into account ([1] p. 50, Equation (1.10.9))

$$\mathcal{L}[x^{\beta-1}E_{\alpha,\beta}(ax^\alpha)](s) = \frac{s^{-\beta}}{1 - as^{-\alpha}},$$

we have

$$\mathcal{L}[u](s) = \frac{1}{\lambda} \frac{b}{s - as^\alpha} \mathcal{L}\left[u\right]\left(\frac{s}{\lambda}\right) + \frac{u(0)}{s - as^\alpha} + \frac{u(0)a}{s^{2-\alpha} - as}. \quad (39)$$

Let $F(s) := \mathcal{L}[u](s)$, and define

$$\theta(s) := \frac{b}{s - as^\alpha}, \quad \varphi(s) := \frac{u(0)}{s - as^\alpha} + \frac{u(0)a}{s^{2-\alpha} - as}.$$

Then (39) can be expressed as

$$F(s) = \frac{1}{\lambda} \theta(s) F\left(\frac{s}{\lambda}\right) + \varphi(s).$$

Iterating this equation n times yields

$$F(s) = F(s\lambda^{-n})\lambda^{-n} \prod_{j=0}^{n-1} \theta(s\lambda^{-j}) + \sum_{k=0}^{n-1} \lambda^{-k} \varphi(s\lambda^{-k}) \prod_{j=0}^{k-1} \theta(s\lambda^{-j}).$$

Letting $n \rightarrow +\infty$, we (formally) obtain

$$F(s) = \sum_{k=0}^{\infty} \lambda^{-k} \varphi(s\lambda^{-k}) \prod_{j=0}^{k-1} \theta(s\lambda^{-j}). \quad (40)$$

We next give sufficient conditions for the convergence of the formal series in (40) and for the validity of the limiting procedure leading to it. Assume that $0 < \lambda < 1$ and take $s > 0$ sufficiently large so that the denominators appearing in $\theta(s\lambda^{-j})$ and $\varphi(s\lambda^{-k})$ do not vanish. Since

$$\theta(s\lambda^{-j}) = \frac{b}{s\lambda^{-j} - a(s\lambda^{-j})^\alpha} = \frac{b\lambda^j}{s - as^\alpha \lambda^{j(1-\alpha)}},$$

and $0 < \alpha < 1$, there exist a constant $C > 0$ and $j_0 \in \mathbb{N}$ such that

$$|\theta(s\lambda^{-j})| \leq C\lambda^j, \quad j \geq j_0.$$

Consequently, after absorbing the finitely many factors corresponding to $0 \leq j < j_0$ into a constant $C_0 > 0$, we obtain

$$\left| \prod_{j=0}^{k-1} \theta(s\lambda^{-j}) \right| \leq C_0 C^k \lambda^{k(k-1)/2}$$

for all sufficiently large k . Moreover, from the definition of φ we have

$$\lambda^{-k} \varphi(s\lambda^{-k}) = \frac{u(0)}{s - as^\alpha \lambda^{k(1-\alpha)}} + \frac{u(0)a}{s^{2-\alpha} \lambda^{-k(1-\alpha)} - as},$$

and hence

$$\lambda^{-k} \varphi(s\lambda^{-k}) = O(1), \quad k \rightarrow \infty.$$

It follows that the absolute value of the k th term in the series (40) is bounded, for all sufficiently large k , by

$$C_1 C^k \lambda^{k(k-1)/2}$$

for some constant $C_1 > 0$. Since

$$\sum_{k=0}^{\infty} C^k \lambda^{k(k-1)/2} < \infty$$

by the ratio test, the series in (40) converges absolutely.

Furthermore, if

$$F(r) = O(r^{-1}), \quad r \rightarrow +\infty,$$

then

$$\lambda^{-n} F(s\lambda^{-n}) = O(1), \quad n \rightarrow \infty.$$

Combining this with the above estimate gives

$$F(s\lambda^{-n}) \lambda^{-n} \prod_{j=0}^{n-1} \theta(s\lambda^{-j}) \rightarrow 0, \quad n \rightarrow \infty.$$

Thus, under these conditions, the limiting procedure leading to (40) is justified in the Laplace-transform domain. If the function defined by (40) admits an inverse Laplace transform with the required regularity, then the inverse transform provides a solution of the integral Equation (38), and hence of Equation (36). A detailed study of the inversion and regularity of this representation is beyond the scope of the present paper.

In particular, when $a = 0$, the series (40) becomes

$$F(s) = u(0) \sum_{k=0}^{\infty} s^{-k-1} b^k \lambda^{k(k-1)/2},$$

and thus

$$u(t) = u(0) \sum_{k=0}^{\infty} b^k \lambda^{k(k-1)/2} \frac{t^k}{k!}.$$

This matches the form of solution (5) in Section 1.

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